Ch. 1 – Introduction:

Units and Measurement

Scientific Notation

- In physics we encounter very large and very small numbers.

- For example:

  - Speed of light: c = 300,000,000 m/s

  - Planck's constant:

    h = .000000000000000000000000000000000663 J·s

- We want a better way to write these numbers.

Scientific Notation

- Numbers in scientific notation are written in this form:

$$a.bc \times 10^{d}$$

- d>0 → decimal point moves to the right to get back to ordinary notation (large numbers)
- d<0 → decimal point moves to the left to get back to ordinary notation (small numbers)
- Examples:
  - Speed of light: $c = 3.00 \times 10^{8}\ \frac{m}{s}$
  - Planck's constant: $h = 6.63 \times 10^{-34} J \cdot \mathrm{s}$
  - Wavelength of green light: $\lambda = 5 \times 10^{-9} m = .000000005\ \mathrm{m}$

Multiplying and Dividing

- Scientific notation also makes multiplying and dividing large and small numbers easier.
- For example:

- $(3.2 \times 10^{3})(2.1 \times 10^{-2}) = (3.2 \times 2.1) \times 10^{3+(-2)}$
  $= 6.72 \times 10^{1} = \boxed{67.2}$

- $\frac{3.2\times10^{3}}{2.1\times10^{-2}} = \left(\frac{3.2}{2.1}\right) \times 10^{3-(-2)} = 1.524 \times 10^{5}$
  $= \boxed{152400}$

Significant digits

- Not all digits returned by your calculator necessarily mean anything
- Measured values are limited in resolution: multiplying or dividing them cannot make them more precise (i.e. accurate to more digits)
- If you had three nonzero digits in the numbers you plugged in, then only three digits of the output are meaningful.
- In general:
  - (number of significant digits out) = (**smallest** number of significant digits among the input values)
- For example:
  - Multiply **(3.23)x(2.4789) = 8.006847**
  - Only the first three digits are significant: **8.01** (3 digits, rounding last digit to nearest value)

Units

- We will primarily use the **metric system**
  - (also known as the **SI** (standard international) or **MKS** (meter-kilogram-second) system)
- Basic units for now:
  - **Meters (m)** for distance or length
  - **Kilograms (kg)** for mass
  - **Seconds (s)** for time
- Later we will add two more basic units:
  - **Kelvin (K)** for temperature
  - **Coulomb (C)** for charge
- All other units we need can be built out of combinations of these

Adjusting unit sizes

- The sizes of our units can be adjusted by adding prefixes.
- The prefixes we will see over the course of the year are:
    - Kilo (k) = 10^3 (one thousand)
    - Centi (c) = 10^{-2} (one hundredth)
    - Milli (m) = 10^{-3} (one thousandth)
    - Micro (µ) = 10^{-6} (one millionth)
    - Nano (n) = 10^{-9} (one billionth)
    - Pico (p) = 10^{-12} (one trillionth)
- For example:

$$10 \text{ kg} = 10{,}000 \text{ g} = 10^7 \,\mu\text{g}$$

$$1 \text{ cm} = 10 \text{ mm} = .01 \text{ m} = 10^{-5} \text{ km}$$

Other Units

- Other (non-MKS) units will occasionally be used, such as:
    - Feet (ft) and miles for distance
    - Pounds (lbs) for weight or force
    - Celsius (°C) for temperature
    - Liters (L) for volume
    - Pascals or atmospheres for pressure
- In general, we will convert these into MKS units before using them.
- All quantities being plugged into an equation **must have consistent units.**
- For example, you can't combine a distance in miles with a distance in meters in the same equation.

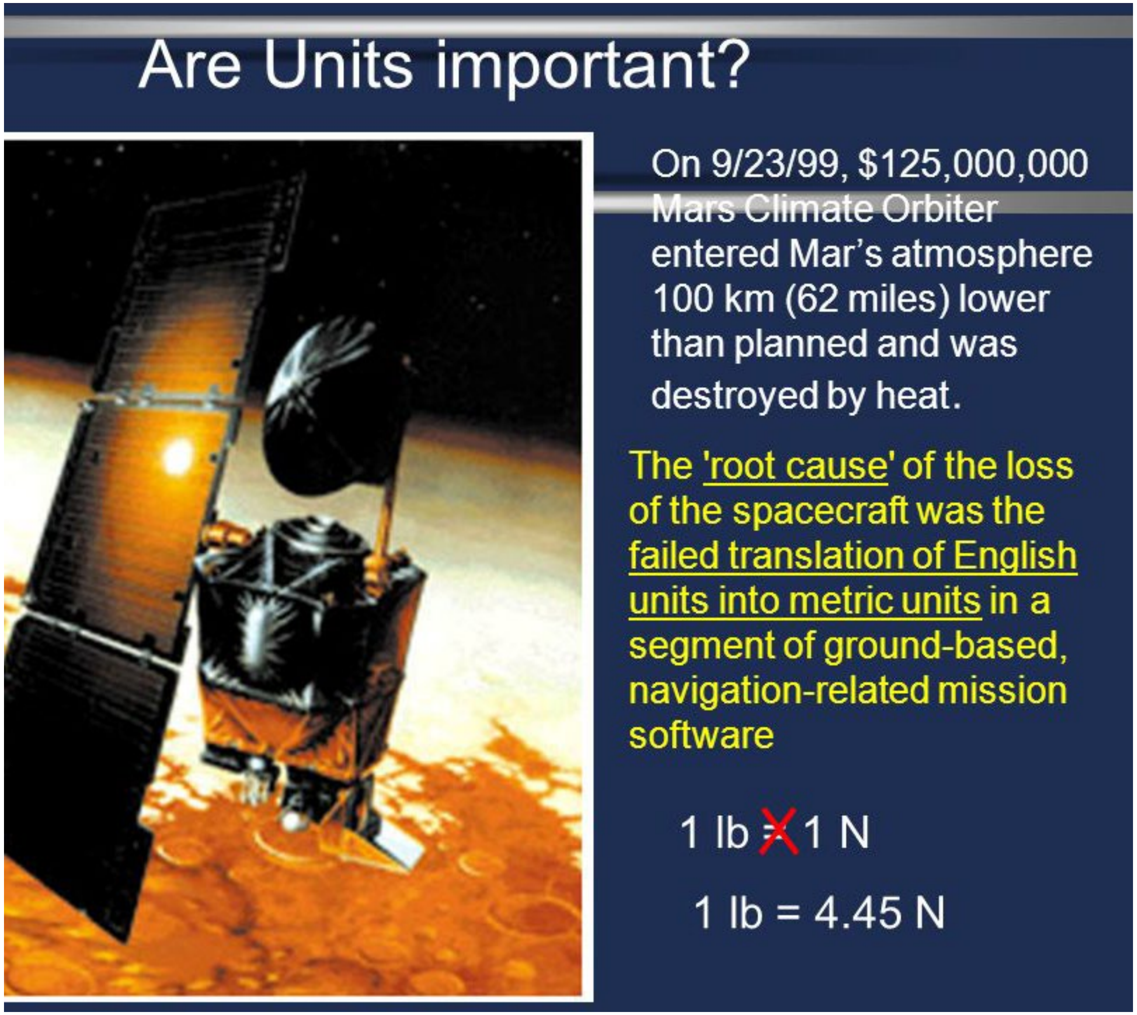
Unit Conversion

- **General strategy:**
  - Look up conversion factor(s) in book or in formula sheet
  - Form the conversion into a ratio and multiply the original quantity by this ratio, so that the unwanted units cancel. (see examples below).
  - Multiple conversion factors may be needed
  - Powers of conversion factors may be needed.

Example: Unit conversion

- 2.5 ft equals how many meters?
  - Conversion factor: 1 m = 3.28 ft
  - Form the ratio $\left(\frac{1m}{3.28\ ft}\right)$. Since the top and bottom are equal, this ratio really just equals 1.
  - Multiply this ratio times the original value and cancel the unwanted units:
  - $(2.5\ ft) \cdot \left(\frac{1m}{3.28\ ft}\right) = \left(\frac{2.5\ \cancel{ft}}{3.28\ \cancel{ft}}\right) \cdot 1\mathrm{m} = \boxed{.762\ \mathrm{m}}$
  - **Note** that units can be multiplied, divided, and canceled, just like numbers.

The importance of unit conversion



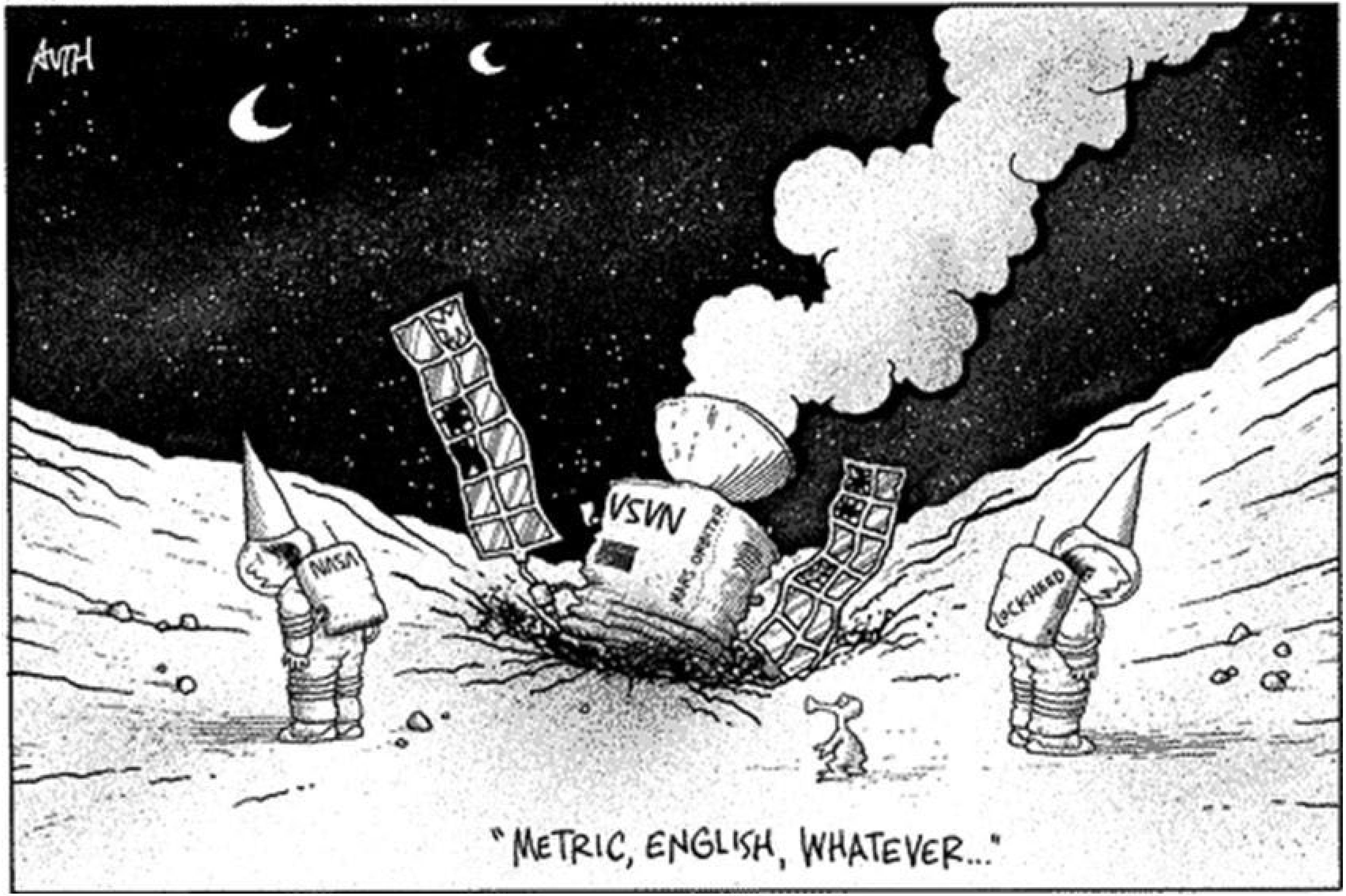
Are Units important?

On 9/23/99, $125,000,000 Mars Climate Orbiter entered Mar's atmosphere 100 km (62 miles) lower than planned and was destroyed by heat.

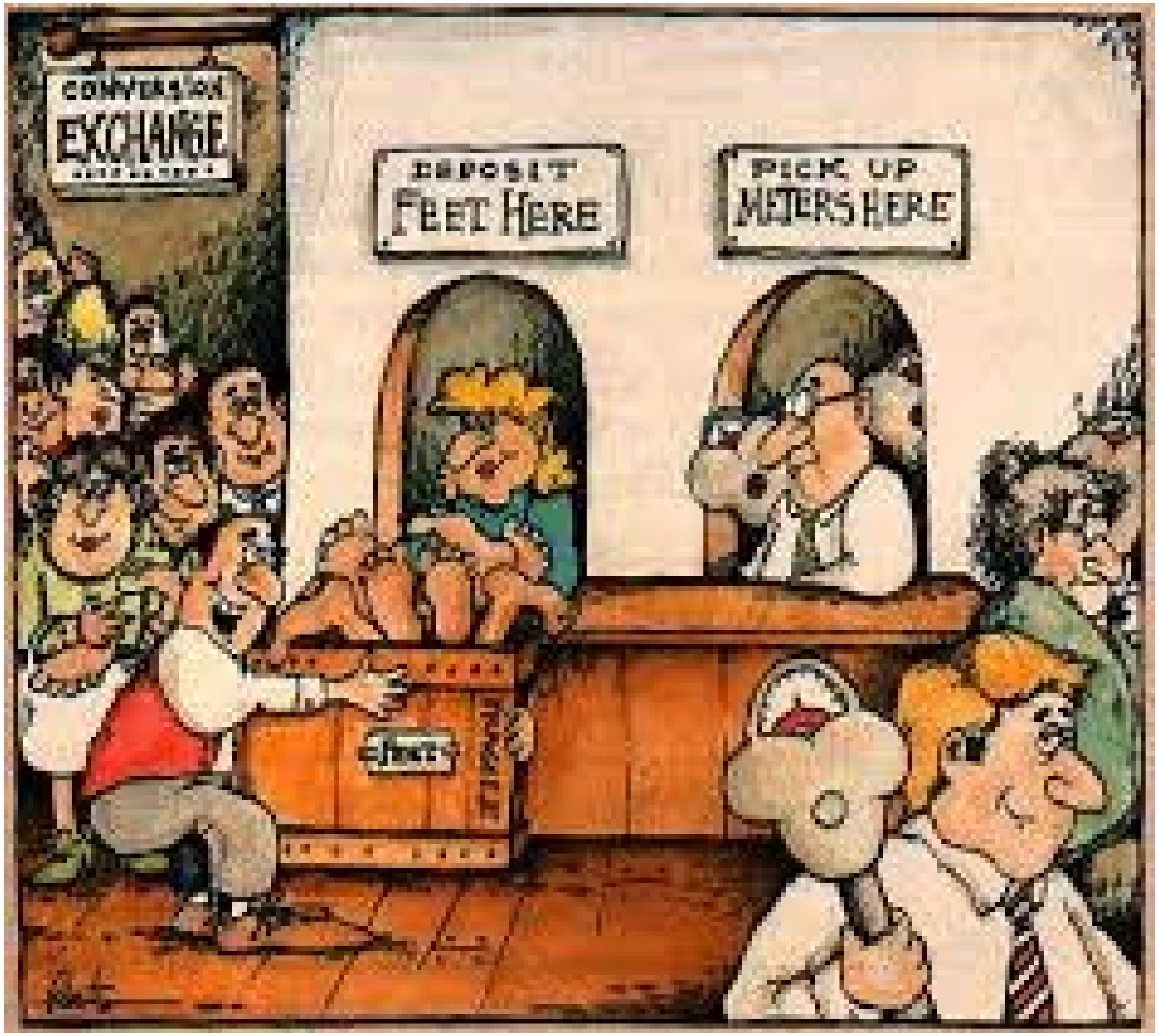
The 'root cause' of the loss of the spacecraft was the failed translation of English units into metric units in a segment of ground-based, navigation-related mission software

$1\ \text{lb} \cancel{=} 1\ \text{N}$

$1\ \text{lb} = 4.45\ \text{N}$


AUTH
NASA
MARS ORBITER
NASA
LOCKHEED
"METRIC, ENGLISH, WHATEVER..."

Remember the Mars Climate Orbiter incident from 1999?



CONVERSION
EXCHANGE
DEPOSIT
FEET HERE
PICK UP
METERS HERE

Fortunately, conversion is easy and convenient

Example: Unit conversion

- Convert 2500 m into miles.

  - Conversion factors: 1 km = .621 mile, 1 km =10^3 m
  - $(2500\ \cancel{m}) \cdot \left(\frac{1\ \cancel{km}}{1000\cancel{m}}\right) \cdot \left(\frac{.621\ miles}{1\ \cancel{km}}\right) = \left(\frac{(2500)(.621)}{10^3}\right) \mathrm{m}ile$

    $= \boxed{1.55 \text{ miles}}$

Example: Unit conversion

- What is 2000 ft/min when converted to m/s?

- Conversion factors: 1 m = 3.28 ft , 1 min = 60 s

- $(2000\ \frac{\cancel{ft}}{\cancel{min}}) \cdot \left(\frac{1\ m}{3.28\ \cancel{ft}}\right) \cdot \left(\frac{1\ \cancel{min}}{60\ s}\right) = \left(\frac{2000}{(3.28)(60)}\right)\frac{m}{s}$

$$= \boxed{10.163\ \frac{m}{s}}$$

Example: Unit conversion

- Convert the **area** 3 ft^2 into m^2.
  - Conversion factor: 1 m = 3.28 ft
  - $(3\ ft^2)\cdot\left(\frac{1\ m}{3.28\ ft}\right)^2 = \boxed{.279\ m^2}$
- Convert the **volume** 3 ft^3 into m^3.
  - $(3\ ft^3)\cdot\left(\frac{1\ m}{3.28\ ft}\right)^3 = \boxed{0.0850\ m^3}$

Supplementary material for use in lab

Describing Data: Mean and Standard Deviation

- Suppose we are interested in some physical variable, x.
- Make n measurements. This results in n data points:
  - $x_1, x_2, x_3, \dots \dots, x_n$
- We want a convenient way of summarizing this data.
- Most common way: give a measure of the **center** of the data (the **mean**) and a measure of the **spread** of the data (the **standard deviation**).
- First, some notation.

Notation: Summation symbol

- Denote a generic data point by x_i, where $i = 1, 2, 3, \ldots, n$
- Define the summation symbol:

  $\Sigma =$ capitol Greek letter Sigma

  → take sum of all values of whatever is to the right

- **Example:** $x_1 = 3, \quad x_2 = 4, \quad x_3 = 2$
  - $\Sigma\, x = x_1 + x_2 + x_3 = 3 + 4 + 2 = 9$
  - $\Sigma\, x^2 = x_1^2 + x_2^2 + x_3^2 = 3^2 + 4^2 + 2^2 = 9 + 16 + 4 = 29$
  - $(\Sigma\, x)^2 = (x_1 + x_2 + x_3)^2 = (3 + 4 + 2)^2 = 9^2 = 81$

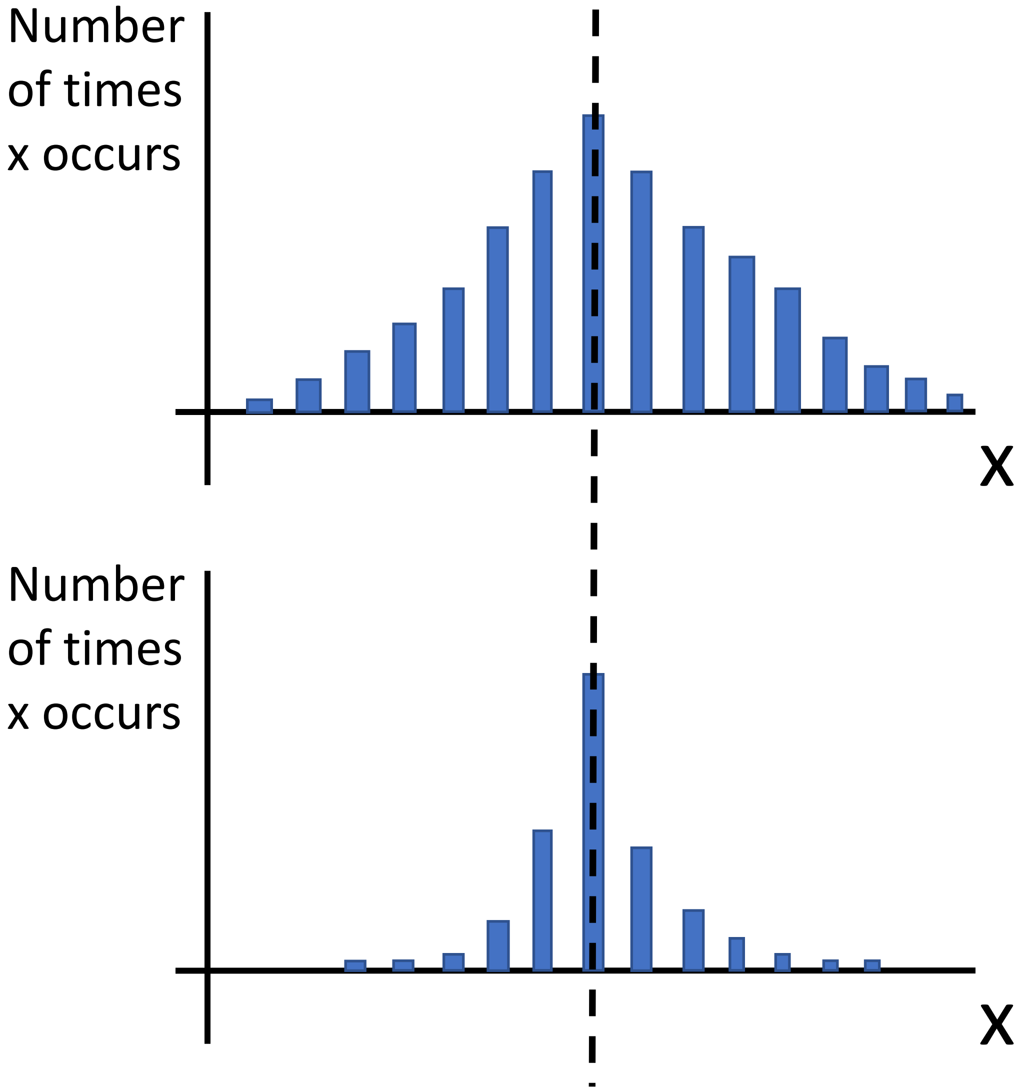
Mean

- The mean or average measures the center of a distribution of data, or equivalently it gives the "most typical" value.
- **Definition:**

$$\boxed{\bar{x} = \frac{1}{n}\sum x = \frac{1}{n}(x_1 + x_2 + x_3 + \ \dots\dots + x_n)}$$

- **Example:** $x_1 = 3,\quad x_2 = 4,\quad x_3 = 2$
- $\bar{x} = \frac{1}{3}(3 + 4 + 2) = \frac{9}{3} = \boxed{3}$

Standard Deviation



- These two data sets have the same mean, but are clearly different.
- In the bottom set, most data points are clustered close to the mean.
- In the top one, many points are far from the mean.
- Clearly the mean tells us more about a typical data point in the bottom set than in the top.
- We need a way to distinguish between these. This will be the **standard deviation**.

Standard Deviation

- The **standard deviation** measures the spread of the data away from the mean, or the width of the distribution.
- Can think of the standard deviation as the average distance of a typical data point from the mean. (This needs to be taken with a grain of salt: see next slide)

- Notation: standard deviation is s (or sometimes σ_{n-1})

- Note we are using the **sample** standard deviation; the **population** standard deviation is slightly different.

Standard Deviation

- Suppose we take literally the statement “standard deviation as the average distance of a typical data point from the mean”. What happens?
- Distance of data point x from the mean: $x - \overline{x}$
- Average distance:

$$\frac{1}{n}\sum(x-\overline{x}) = \frac{1}{n}\sum x - \frac{1}{n}\sum \overline{x} = \frac{1}{n}\sum x - \frac{1}{n}(n\overline{x})$$

$$= \overline{x} - \overline{x} = 0$$

- So we make two changes:
  - We square before averaging, to prevent positive and negative values from canceling.
  - For technical reasons we divide by n-1, rather than n. (One value is not independent once the mean is fixed.)

Standard Deviation

- Definition of (sample) standard deviation:

$$\boxed{s = \sqrt{\frac{1}{n-1}\sum(x-\overline{x})^2}}$$

- The square root is needed to get back to the original units.
- An alternate (equivalent) expression is often useful:

$$\boxed{s = \sqrt{\overline{x^2} - \overline{x}^2}}$$

Order of operations is important: In first term, square then average. In second term, average, then square.

Standard error

- In labs you may be asked to find the **standard error** or **margin of error:**

$$\delta x = \frac{s}{\sqrt{n}}$$

- δx gives a measure of how well the mean is to be trusted: if it is small, the **estimated mean** (the sample mean) from the data is likely to be close to the **true mean** (the population mean).

- The outcome of an experiment is often reported as a **confidence interval**: $\overline{x} \pm \delta x$

- This indicates that the true mean is likely to be within distance δx of the estimated sample mean $\overline{x}$

Example

- A voltage is measured n = 5 times. The measured values are: $\mathrm{V} = 10V, 12V, 11V, 9V, 11V$
- Form a table: Fill data in, then take sum of column in last row.

$\mathbf{V}$	$\mathbf{V}-\overline{\mathbf{V}}$	$(\mathbf{V}-\mathbf{V})^2$
10	-.6	.36
12	1.4	1.96
11	.4	.16
9	-1.6	2.56
11	.4	.16
53	0	5.2

$\Sigma\,\mathrm{V}$ $\quad\Sigma\,(\mathrm{V}-\overline{V})$ $\quad\Sigma(\mathrm{V}-\overline{V})^2$

$$\overline{V} = \frac{1}{n}\Sigma V = \frac{1}{5}\,(53) = \boxed{10.6\ V}$$

$$s = \sqrt{\frac{1}{n-1}\sum\left(V-\overline{V}\right)^2}$$

$$= \sqrt{\frac{5.2}{4}} = \boxed{1.14\ V}$$

$$\delta V = \frac{s}{\sqrt{n}} = \frac{1.14}{\sqrt{5}} = \boxed{.51\ V}$$

Measurement uncertainty

- In addition to the **statistical uncertainty** (standard error) due to multiple measurements, there is also **measurement uncertainty** due to limited resolution of the measuring device.
- For example if measuring a length with a ruler marked in multiples of ¼ inch, the minimum resolution is 1/8 inch.
- In general, minimum measurement error is ½ spacing between markings on measurement device.
- Any decimal places beyond this resolution limit are meaningless.
- Multiple sources of error are added as follows:

$$
\begin{aligned}
&\mathit{total\ error}\\
&= \sqrt{(\mathit{statistical\ error})^2 + (\mathit{measurement\ error})^2 + \ \dots\dots}
\end{aligned}
$$

Measurement uncertainty

- Often error is expressed as **percent error** or **percent difference**.
- The **percent error** from an accepted value is:

$$\%\ error = \frac{|accepted\ value\ - measured\ value|}{accepted\ value} \times 100\%$$

- **Example**: An experiment measures a value of 1.7x10^{-19} C for the charge of an electron. What is its percent error from the accepted value of 1.6x10^{-19} C ?

$$\%\ error = \frac{|1.7\text{x}10{-}19 - 1.6\text{x}10{-}19|}{1.6\text{x}10{-}19} \times 100\% \ = \boxed{6.25\%}$$

Measurement uncertainty

- The **percent difference** between two measured values is:

$$\% \, diff. = \frac{|value\ 1 - value\ 2|}{\left(\frac{value\ 1 + value\ 2}{2}\right)} \times 100\%$$

- **Example**: Two experiments measure values of 1.7x10^{-19} C and 1.8x10^{-19} C for the charge of an electron. What is its percent difference between these values?

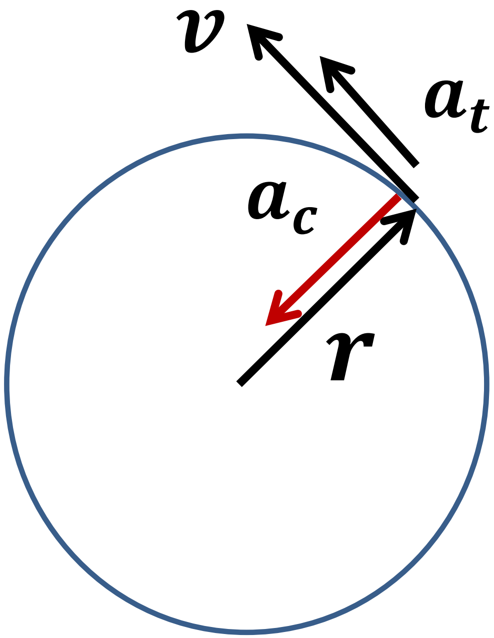
$$\% \, error = \frac{|1.7\text{x}10{-}19 - 1.8\text{x}10{-}19|}{\left(\frac{1.7\text{x}10{-}19 + 1.8\text{x}10{-}19}{2}\right)} \times 100\%$$

$$= \frac{|1.7\text{x}10{-}19 - 1.8\text{x}10{-}19|}{1.75\text{x}10{-}19} \times 100\% \quad = \boxed{5.71\%}$$

CH. 10 –ROTATIONAL MOTION

Rotational Motion

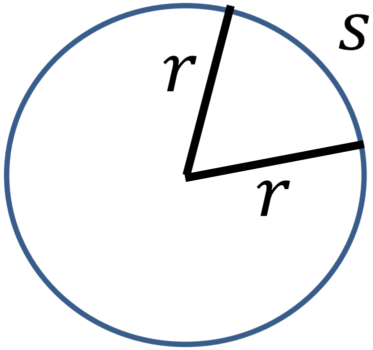
- Suppose object is rotating in a circle of radius r.
- It could be a small object going in a circle about some fixed point (**orbital motion**) or a large object rotating about its center (**revolution**).
- If speed is constant, only acceleration is **centripetal** (toward center): $a_c = \frac{v^2}{r}$.
- If speed is changing, there is also **tangential** acceleration, parallel to velocity: $a_t = \frac{dv}{dt}$.


$\boldsymbol{v}$
$\boldsymbol{a_t}$
$\boldsymbol{a_c}$
$\boldsymbol{r}$

Rotational Quantities

- **Period:** $T =$ time for one revolution or rotation
- **Frequency:** $f = \frac{1}{T} =$ # of revs. per unit time

  (in Hertz: 1 Hz = 1 rev./sec)
- **Angle and arclength:** $\theta = \frac{s}{r}$ $(\theta \text{ in radians})$



Rotational Quantities

- **Linear motion** - we have three main kinematic variables:
  - Position: x (in m)
  - Velocity: $v = \frac{dx}{dt}$ (in m/s)
  - Acceleration: $a = \frac{dv}{dt}$ (in m/s^2)

We want analogous variables to describe rotational or angular motion.

Rotational Quantities

- Rotational analogs of kinematic variables:
  - – Angle: θ (in rad)
  - – **Angular velocity**: $\omega = \frac{d\theta}{dt}$ (in rad/s)
  - – **Angular acceleration**: $\alpha = \frac{d\omega}{dt}$ (in rad/s^2)
- ω and α are vectors, but we generally will treat them as scalars since we will only look at situations where their directions are fixed.
- More greek letters: $\omega =$ omega and $\alpha = alpha$

Rotational vs. Linear Quantities

- x, v, and a have a simple relation to θ, ω, and α.
- Since $\theta = \frac{s}{r}$, we have:

$$\omega = \frac{d\theta}{dt} = \frac{d(\frac{s}{r})}{dt} = \frac{1}{r}\frac{ds}{dt} = \frac{v}{r}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d(\frac{v}{r})}{dt} = \frac{1}{r}\frac{dv}{dt} = \frac{a_t}{r}$$

- So:

$$\boxed{\begin{aligned} \omega &= \frac{v}{r} \\ \alpha &= \frac{a}{r} \end{aligned}}$$

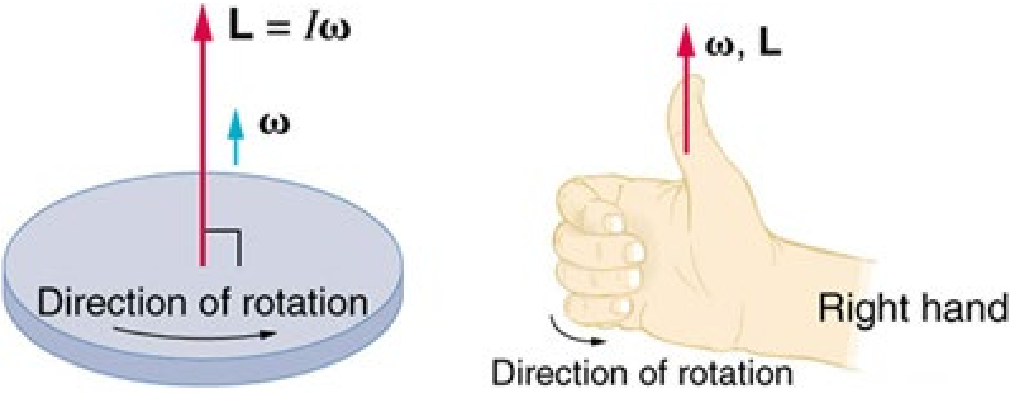
For the remainder of the chapter, a_t will simply be written as a.

Some Technicalities

- **Signs**:
  - $\omega > 0$ means counter-clockwise rotation (θ increasing)
  - $\omega < 0$ means clockwise rotation (θ decreasing)
- **Radians:** radians are not real units, but fractions of a circle. They can be added or dropped as needed to make the units work out.
- For example: notice that $\omega = \frac{v}{r}$ the radians have to be dropped on the left side make the units match on both sides.

ω as vector

- Normally we will treat angular speed as a scalar and indicate direction by a sign.
- But in reality, it is a vector: the direction is perpendicular to the plane of rotation in the direction given by the **right-hand rule**.


L = *I*ω
ω
Direction of rotation
ω, L
Right hand
Direction of rotation

(a) (b)

Fingers: direction of rotation
Thumb: direction of ω

Constant angular acceleration

- In direct analogy to the constant acceleration equations of chapter 2, we can write down **constant angular acceleration equations**:

$$\boxed{\begin{aligned}\theta &= \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \\ \omega &= \omega_0 + \alpha t \\ \omega^2 &= {\omega_0}^2 + 2\alpha(\theta - \theta_0)\end{aligned}}$$

Compare to:

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$v = v_0 + at$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

- We will continue to extend the list of analogies throughout this chapter.

Example 1

- A bicycle has tires of radius r=.35 m. Each is turning at a **constant** rate of 4 rev/s (i.e. $f = 4\ Hz$).

(a) How fast does a point on the rim of the tire move?

$$\omega = 2\pi f = 2\pi(4\ Hz) = 25.13\ rad/s$$

$$v = r\omega = (.35m)\left(25.13\frac{rad}{s}\right) = \boxed{8.80\ m/s}$$

(b) How far does a point on the rim travel in 2 seconds?

$$\theta = \omega t = \left(25.13\frac{rad}{s}\right)(2\ s) = \ 50.26\ rad$$

$$x = r\theta = (.35\ m)(50.26\ rad) = \boxed{17.6\ m}$$

(or use x=vt)

Example 1 (continued)

Suppose now that the bicyclist pushes harder on the pedal, causing a **constant** angular acceleration of $\alpha = 5\frac{rad}{s^2}$.

Now the values of v and ω from the before are the ***initial*** values.

(c) What are the centripetal and tangential accelerations at a point on the tire's rim at t=0?

$$a_c = \frac{{v_0}^2}{r} = \frac{(8.8)^2}{.35} = \boxed{221.26\, m/s^2}$$

$$a_t = r\alpha = (.35)(5) = \boxed{1.75\, m/s^2}$$

- Note that a_t is constant, since α is constant, but that a_c changes over time as the bike accelerates.
- a_t is also the rate of linear acceleration of the bike.

Example 1 (continued)

(d) Find v and ω at t=2 s after acceleration begins.

$$\omega = \omega_0 + \alpha t = 25.13 + (5)(2) = \boxed{35.13\, rad/s}$$

$$v = r\omega = (.35m)\left(35.13\frac{rad}{s}\right) = \boxed{12.3\; m/s}$$

(Check using $v = v_0 + a_t t$)

(e) Find the angle through which the wheel turns during these two seconds.

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2 = (25.13)(5) + \frac{1}{2}(5)(2)^2 = \boxed{60.26\, rad}$$

(Check using $\omega^2 = {\omega_0}^2 + 2\alpha(\theta - \theta_0)$)

Example 1 (continued)

(f) How long does it take for the wheel to reach angular speed of 50 rad/s?

$$\omega = \omega_0 + \alpha t \quad \rightarrow \quad t = \frac{\omega - \omega_0}{\alpha} = \frac{50 - 25.13}{5}$$

$$= \boxed{4.97\ s}$$

Example 2

- A DVD drive rotates a disc at angular speed 31.4 rad/s.

(a) Find the rotational frequency, f.

$$f = \frac{\omega}{2\pi} = \frac{31.4\, rad/s}{2\pi} = \boxed{5.0\, Hz}$$

(b) Find the linear speed of a point on the disc at distance r=.041 m from the center.

$$v = r\omega = (.041)(31.4) = \boxed{1.29\, m/s}$$

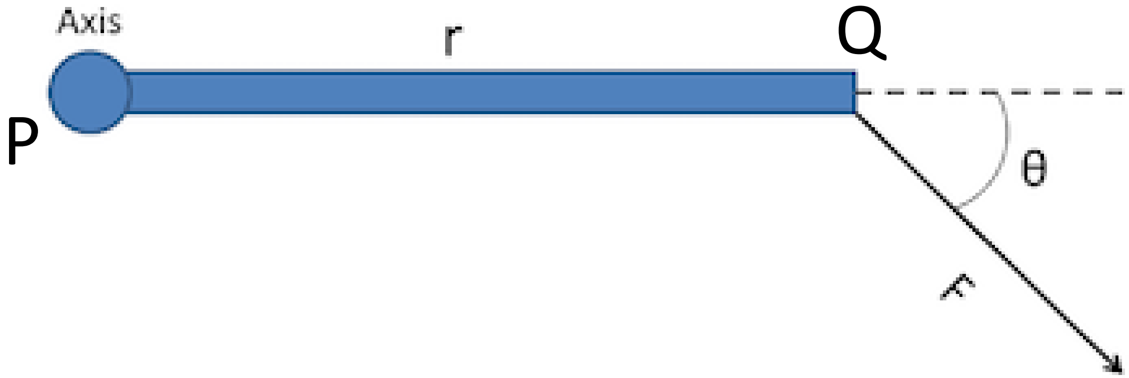
(c) Suppose disk reaches this speed (from rest) in time .006 s. Find the angle through which the disk turns.

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2 = (0)(.006) + \frac{1}{2}(5233.3)(.006)^2$$

$$= \ 0.94\, rad \ = \boxed{5.4°}$$

Torque

- An object is free to rotate about some axis, through **pivot point** P.
- Imagine a force $\boldsymbol{F}$ acting at point Q.
- $r =$ distance from P to Q
- $m =$ mass of object
- θ is the angle between $\boldsymbol{F}$ and **r**

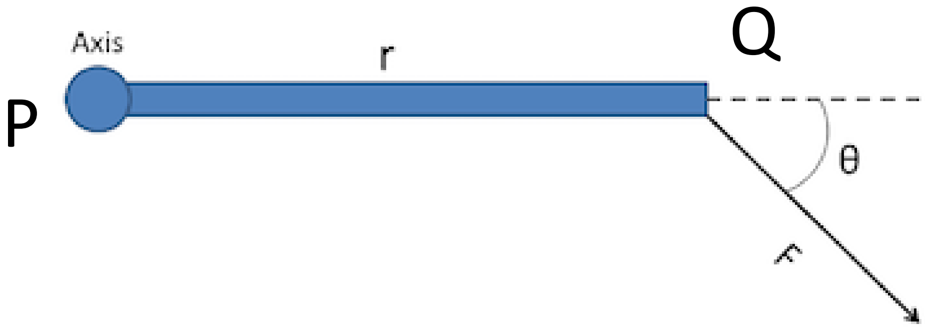


Torque

- The rotational analog of force will be **torque**:

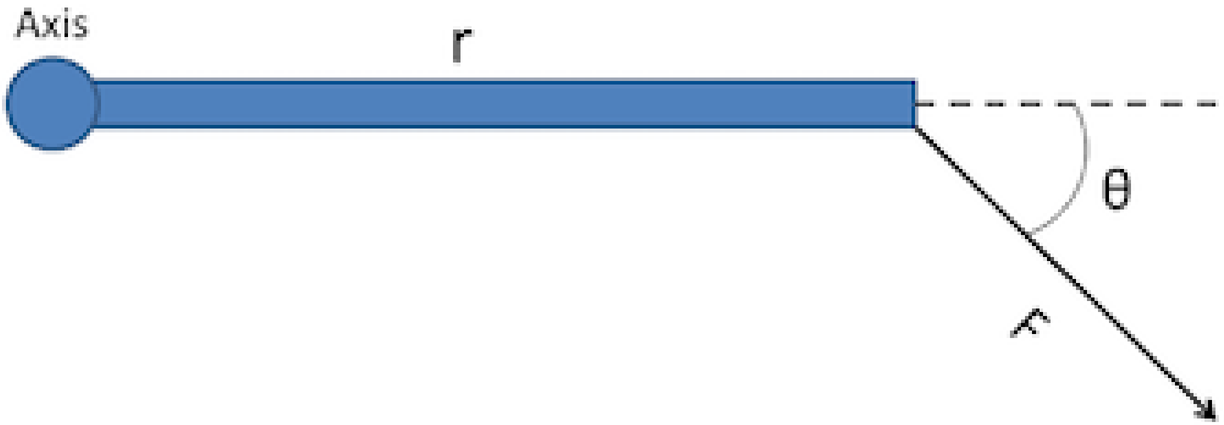
$$\boxed{\tau = Fr\sin\theta}$$

- This is actually the magnitude of the torque vector. The full vector is: $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$
- Units of torque: N · m (or ft · lb)
- Note that the torque depends not only on the force, but also on *where* the force acts



Torque

$$\boxed{\tau = Fr\sin\theta}$$



- Torque is positive if the force causes a counter-clockwise rotation
- It is negative if it causes clockwise rotation.

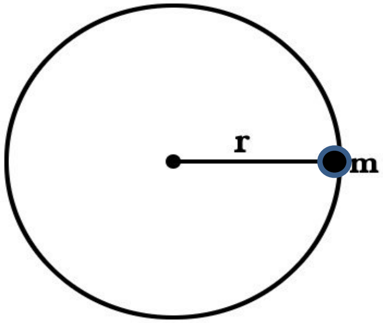
Moment of Inertia

- The rotational analog of mass is **moment of inertia**, I.
- I is a scalar, with units $kg \cdot m^2$.
- I measures resistance to rotation. It should clearly depend on mass.
- But even if they have the same mass, larger objects are harder to rotate than smaller objects.
- I will depend on: **mass**, **size**, and **shape** of object.

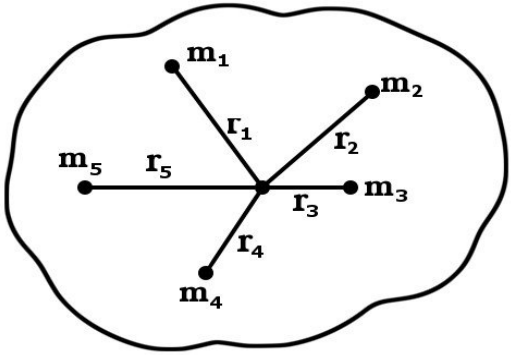
Moment of Inertia

- A **point mass** is an object small enough that we can ignore its size compared to r, the distance from axis
- For a point mass: $\boxed{I = mr^2.}$
- For multiple point masses, just add them:

$$I = \sum m_i r_i^2$$
$$= m_1 r_1^2 + m_2 r_2^2$$
$$+ m_3 r_3^2 + \cdots$$



$I = mr^2$



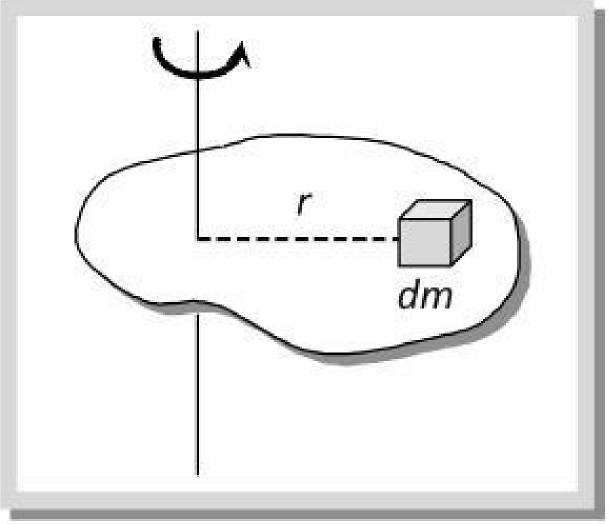
$I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2 + m_5 r_5^2$

Moment of Inertia

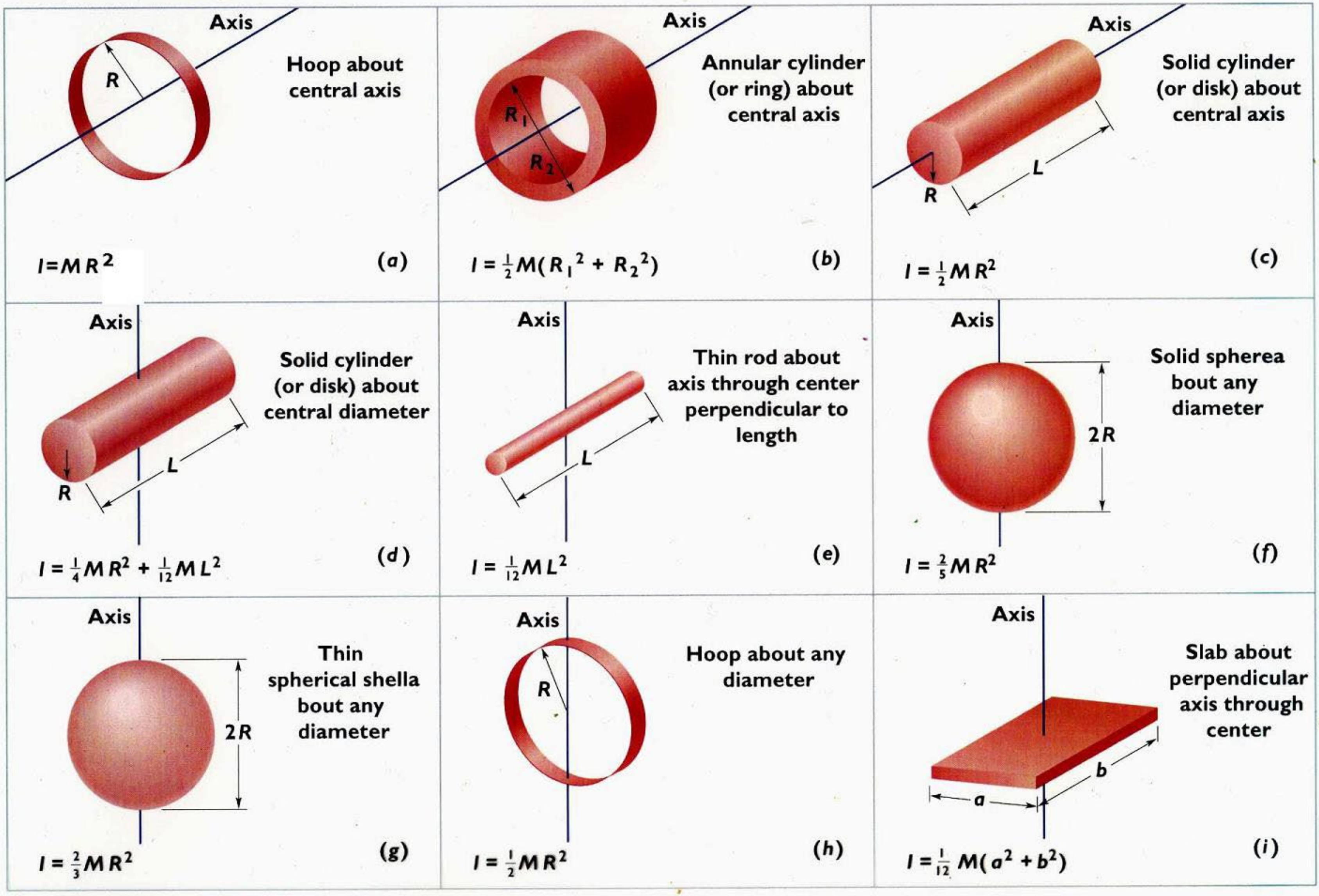
- For a large object, you need to integrate to find the moment of inertia:

$$\boxed{I = \int r^2 \, dm}$$

- Different points in the object are at different values of r. The integral essentially computes the average value of r^2.
- Fortunately other people have computed the integral for common shapes, and we can just steal their results



Moment of Inertia


Axis
Hoop about central axis
$I = MR^2$
(a)
Axis
R_1
R_2
Annular cylinder (or ring) about central axis
$I = \frac{1}{2}M(R_1^2 + R_2^2)$
(b)
Axis
Solid cylinder (or disk) about central axis
L
R
$I = \frac{1}{2}MR^2$
(c)
Axis
Solid cylinder (or disk) about central diameter
L
R
$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$
(d)
Axis
Thin rod about axis through center perpendicular to length
L
$I = \frac{1}{12}ML^2$
(e)
Axis
Solid spherea bout any diameter
$2R$
$I = \frac{2}{5}MR^2$
(f)
Axis
Thin spherical shella bout any diameter
$2R$
$I = \frac{2}{3}MR^2$
(g)
Axis
Hoop about any diameter
R
$I = \frac{1}{2}MR^2$
(h)
Axis
Slab about perpendicular axis through center
b
a
$I = \frac{1}{12}M(a^2 + b^2)$
(i)

Newton's 2nd law: Rotational Version

- Torque measures the ability of a force to cause rotation.
- Moment of inertia measures resistance to rotation.
- Together they determine the angular acceleration:

$$\boxed{\Sigma\tau = I\alpha}$$

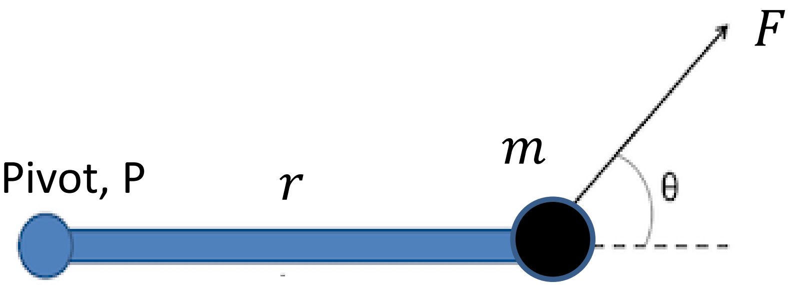
Some additional points

- Both τ and I depend on the location of the pivot point P, but they should still obey $\Sigma\tau = I\alpha$ regardless of how P is chosen.
- In some problems there is a clear rotation axis, and then that is the natural point to take as P.
- In other problems, the choice of P is arbitrary, so you can choose whatever point makes the problem simplest.
- Choose P to be the point where the largest number of forces act, since the torques for those forces will vanish there (since r=0).

Example 3

- An object of mass $m = 2\ kg$ is attached to the end of a light (negligible mass) rod of length $r = 3\ m$, pivoted at the other end. A force of size $F = 1\ N$ is applied to the mass at an angle of $\theta = 30°$.
- The angle of the force doesn't change as the rod rotates, and the mass starts from rest.
- Through what angle does the mass and rod rotate in the first 5 seconds?

The mass is small compared to , so treat as point mass.



Example 3 (continued)

- Point mass: $I = mr^2 = (2)(3)^2 = 18\,kg \cdot m^2$
- Torque: $\tau = rF\sin\theta = (3)(1)\sin 30° = 1.5N\,m$
  - Note that τ is positive: force is causing counter-clockwise rotation
- Angular acceleration:

$$\alpha = \frac{\tau}{I} = \frac{1.5}{18} = .083\,rad/s^2$$

- Angle:

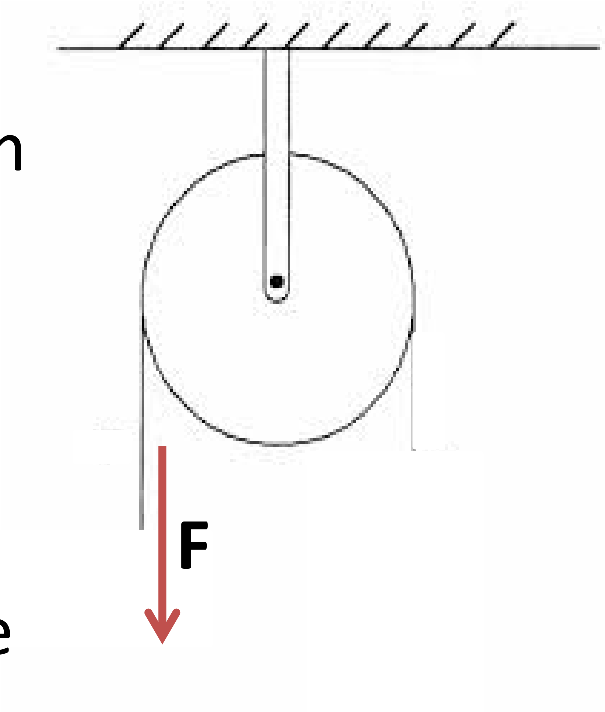
$$\varphi = \varphi_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= 0 + 0 + \frac{1}{2}(.083)(5)^2 = \boxed{1.04\,rad}$$

The rotation angle was called to avoid confusing it with the angle of the force.

Example 4

- A rope is wrapped several times around a uniform cylinder of r=0.1 m and m=50 kg.
- The cylinder is free to rotate around its center.
- The rope hangs down on the left side, and someone pulls with a force of F=20 N.



Example 4 (continued)

(a) What is α?

$\frac{1}{2}$ because mass is cylinder

$$I = \frac{1}{2}mr^2 = \frac{1}{2}(50)(.1)^2 = .25\ \ kg \cdot m^2$$

$$\tau = rF\sin\theta = (.1)(20)\sin 90^\circ = 2.0\ N \cdot m$$

$$\alpha = \frac{\tau}{I} = \frac{2}{.25} = \boxed{8\ rad/s^2}$$

Example 4 (continued)

(b) How fast is a point on the rope moving after 2 s?

$$a = r\alpha = (.1)(8) = .8\, m/s^2$$

$$v = v_0 + at = 0 + \left(.8\,\frac{m}{s^2}\right)(2\ s) = \boxed{1.6\,\frac{m}{s}}$$

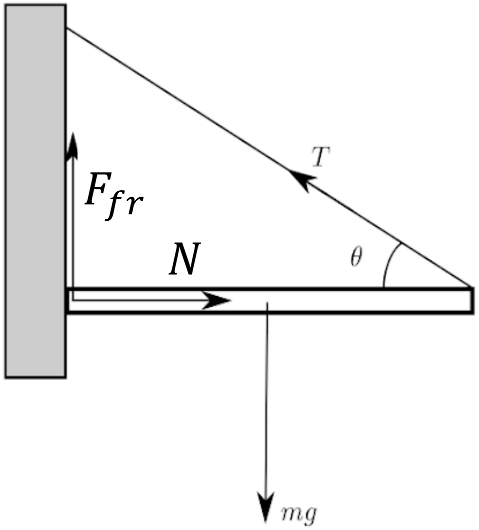
-- OR --

$$\omega = \omega_0 + \alpha t = 0 + \left(.8\,\frac{rad}{s^2}\right)(2\ s) = \quad 16\,\frac{rad}{s}$$

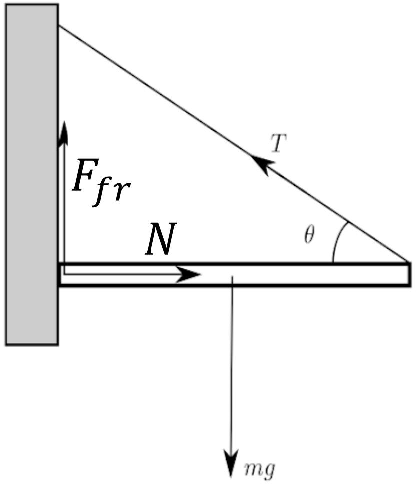
$$v = r\omega = (.1)(16) = \boxed{1.6\, m/s}$$

Example 5

- Mass and length of rod are m and L.
- Angle of cable from rod is θ.
- Left end of rod rests against wall (not attached)
- Find the magnitudes of the friction, normal, and tension forces.



Example 5 (continued)



- $\Sigma F_x = ma_x \rightarrow N - T\cos\theta = 0$

$$N = T\cos\theta$$ Ⓐ

- $\Sigma F_y = ma_y \rightarrow F_{fr} + T\sin\theta - mg = 0$

$$F_{fr} = mg - T\sin\theta$$ Ⓑ

- $\Sigma\tau = I\alpha \ \rightarrow TL\sin\theta - mg\left(\frac{L}{2}\right)\sin 90° = 0$

$$\boxed{T = \frac{mg}{2\sin\theta}}$$ Ⓒ

- Use eqs. Ⓐ, Ⓑ, Ⓒ to find other 2 forces:

Example 5 (continued)

- Plug Ⓒ into Ⓑ:

$$F_{fr} = mg - \frac{mg}{2\sin\theta}\sin\theta = mg - \frac{mg}{2}$$

$$\boxed{F_{fr} = \frac{mg}{2}}$$

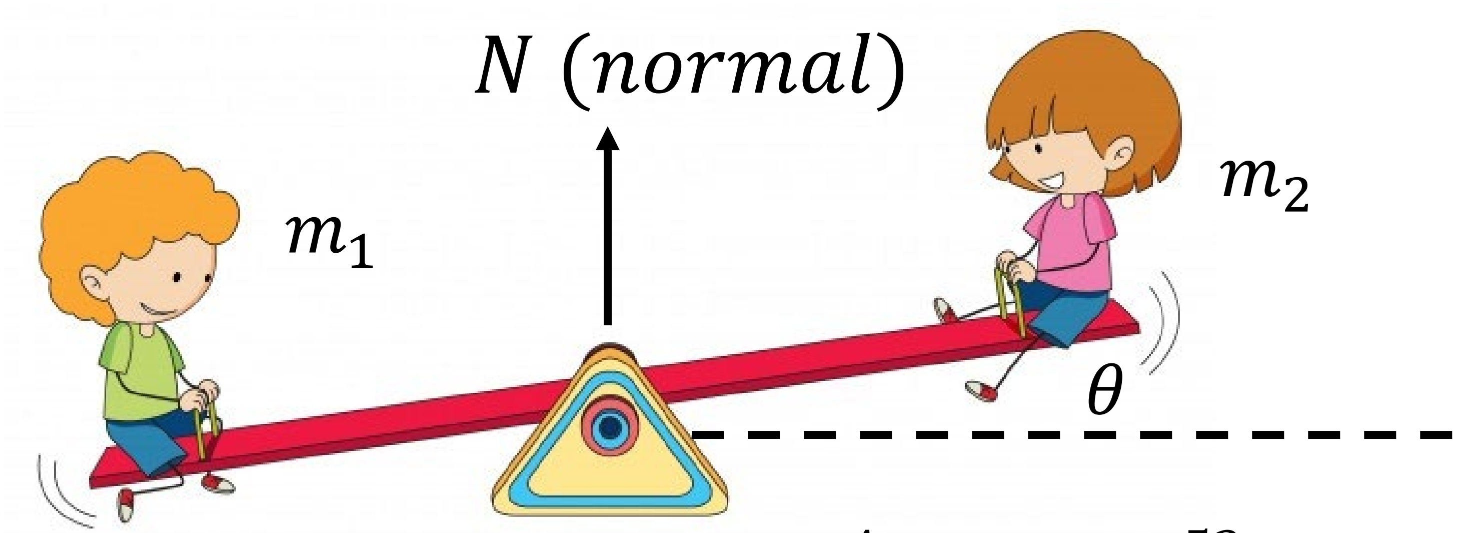
- Plug Ⓒ into Ⓐ:

$$N = \frac{mg}{2\sin\theta}\cos\theta = \frac{mg}{2\tan\theta}$$

$$\boxed{N = \frac{mg}{2\tan\theta}}$$

Example 6

- Given:
  - $m_1 = 1.5m_2$
  - $r_2 = .8\ m$
- If they sit at rest, find: $r_1 = ?$ and $N = ?$


$N\ (normal)$
m_1
m_2
θ

$Answer\!: r_1 = .53\ m$

N= 2.5 m_2 g

Rotational Kinetic Energy

- Recall that for **translational** motion, the kinetic energy is $K = \frac{1}{2}mv^2$.
- But **rotational** motion also takes up kinetic energy. By direct analogy: $K = \frac{1}{2}I\omega^2$.
- In general, both terms may be present at once:

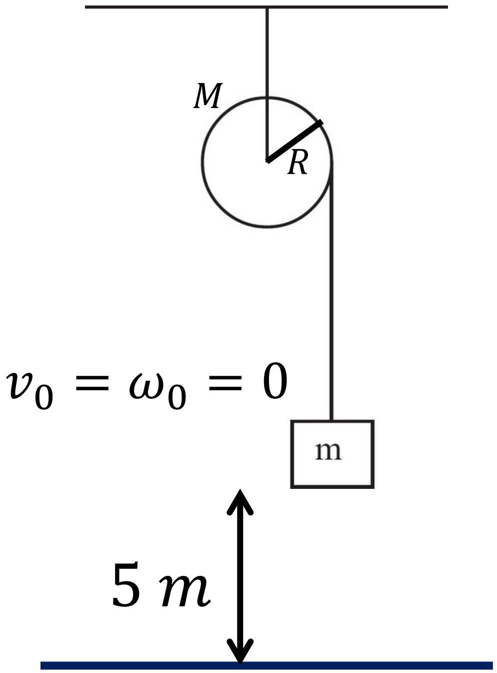
$$\boxed{K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2}$$

Example 7

- A cylindrical pulley has radius R=.5 m and mass M=10 kg. A mass m=3kg is hanging from a rope wound around the cylinder. It is released from rest from a height of $y_0 = 5\,m$.
- What is its angular speed of the pulley

  when m hits the ground?

- Hint: there is no friction - use energy conservation



Example 7 (continued)

$$U_0 + K_0 = U + K \quad \rightarrow \quad mgy_0 + 0 = 0 + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

But $I = \frac{1}{2}MR^2$ for a cylinder and $v = R\omega$, so:

$$mgy_0 + 0 = 0 + \frac{1}{2}m(R\omega)^2 + \frac{1}{4}MR^2\omega^2$$

$$\omega = \sqrt{\frac{2mgy_0}{\left(m + \frac{1}{2}M\right)R^2}} = \sqrt{\frac{2(3)(9.8)(5)}{(8)(.5)^2}} = \boxed{12.1\ m/s}$$

We'll see more examples with rotational kinetic energy in the next chapter.

Translational vs. rotational motion

Translation	Rotation
$x, v = {}^{dx}/_{dt}, a = {}^{dv}/_{dt}$	$\theta, \omega = {}^{d\theta}/_{dt}, \alpha = {}^{d\omega}/_{dt}$
$x = x_0 + v_0 t + \frac{1}{2} a t^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = {\omega_0}^2 + 2\alpha(\theta - \theta_0)$
F, m	τ, I
$\Sigma F = ma$	$\Sigma\tau = I\alpha$
$K = \frac{1}{2}\mathrm{m}v^2$	$K = \frac{1}{2}\mathrm{I}\omega^2$
$p = mv$	$L = I\alpha$

Ch. 11-Rolling, Torque and Angular Momentum

More rotational quantities

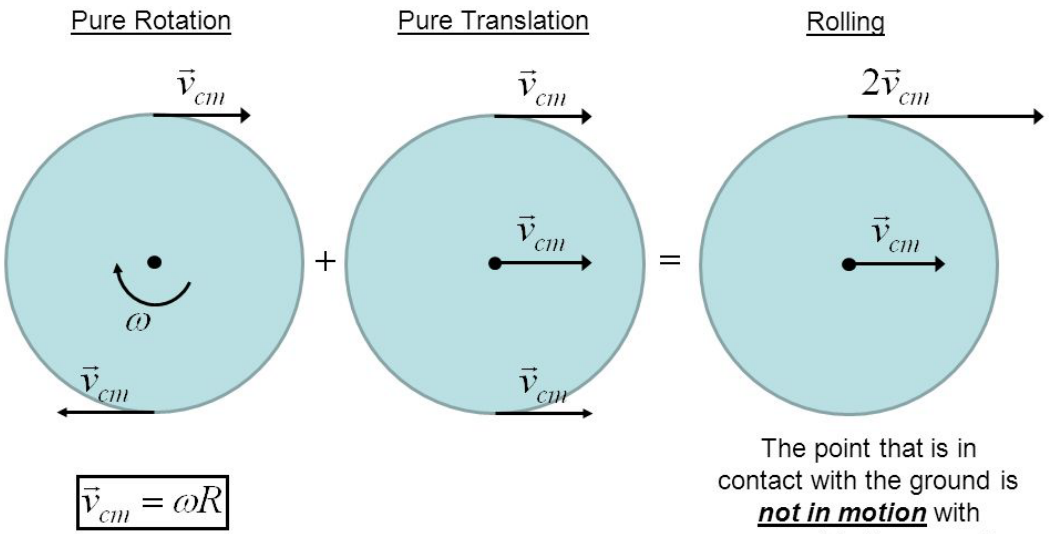
- Continuing with the translational and rotational analogies, this chapter introduces more variable:
  - **Angular momentum** = analog of linear momentum
- One other matter first →

Rolling & Sliding

- **Rotation:** Center of rotation remains fixed. No translation.
- **Rolling** consists of rotation plus translation.

Rolling

One way to view rolling is as a combination of pure rotation and pure translation.


Pure Rotation
Pure Translation
Rolling
$\vec{v}_{cm}$
ω
+
=
$2\vec{v}_{cm}$

$\boxed{\vec{v}_{cm} = \omega R}$

The point that is in contact with the ground is ***not in motion*** with respect to the ground!

Rolling & Sliding

- Rolling and sliding are very different:
  - **Rolling** (without sliding):
    - Friction is **static**
    - Linear and angular speed are proportional, $v = r\omega$
  - **Sliding:**
    - Friction is **kinetic**
    - $\omega = 0$, but v is nonzero
- Can also have rolling with sliding: both occur at once (thing of car wheel skidding on icy road), with no relation between v and ω $(v \neq r\omega)$

Angular Momentum

- **Angular momentum:** $\boxed{L = I\omega}$ (in kg m^2/s)
- This is really a vector equation, $\boldsymbol{L} = I\boldsymbol{\omega}$, with $\boldsymbol{L}$ and $\boldsymbol{L}$ pointing along rotation axis, but we'll treat it as a scalar.
- Just as linear momentum is related to force (**F** = d**p**/dt), angular momentum is related to torque:

$$\boxed{\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt}}$$

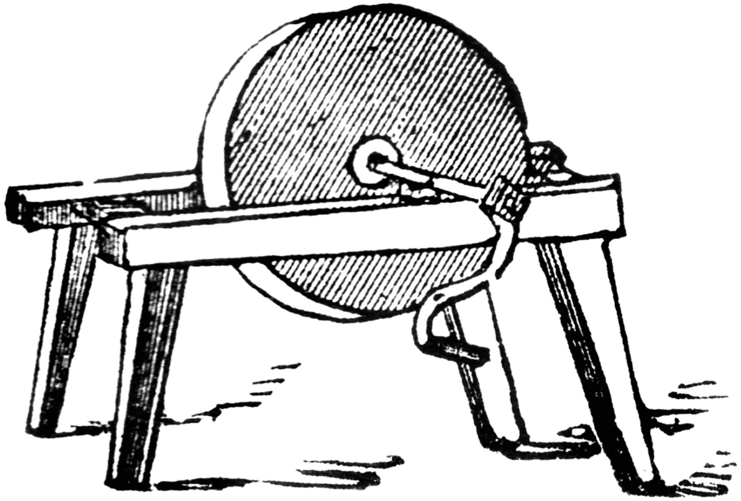
Rotational version of Newton's second law

Conservation of Angular Momentum

- Recall that when there are no external forces then linear momentum is conserved.
- Similarly, angular momentum is conserved when there are no external torques.
- If $\Sigma\tau = 0$, then L is conserved: $\boxed{I_1\omega_1 = I_2\omega_2}$

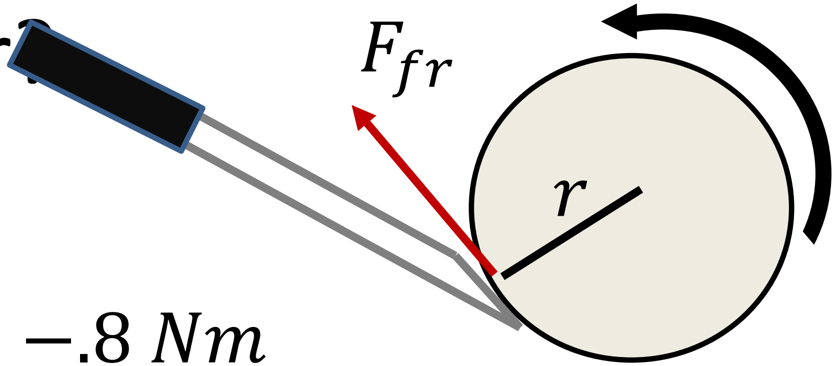
Example 1

- A solid, cylindrical grindstone has mass m=9 kg and radius r=.1 m. It is made to rotate, then a blade is brought into contact with it to be sharpened. Initially it is rotating at $\omega_0 = 188.5\frac{rad}{s}$.



Example 1 (continued)

(a) A blade is brought against the wheel, exerting a friction force of 8N. What will ω be 6 seconds later?



$$I = \frac{1}{2}MR^2 = .045\ kg\ m^2$$

$$\tau = -F_{fr}r = -(8N)(.1m) = -.8\ Nm$$

$$\alpha = \frac{\tau}{I} = -\frac{.8}{.045} = -17.78\ rad/s^2$$

$$\omega = \omega_0 + \alpha t = 188.5 - (17.75)(6) = \boxed{82\ rad/s}$$

Example 1 (continued)

(b) What will L be 6 seconds later?

$$L = I\omega = (.045)(82) = \boxed{3.68\ kgm^2/s}$$

or:

$$L_0 = I\omega_0 = (.045)(188.5) = \ 8.48\ kgm^2/s$$

$$\tau = \frac{\Delta L}{\Delta t} \ \rightarrow \ L = L_0 + \Delta L = \ L_0 + \tau\Delta \mathrm{t} = \boxed{3.68\ kgm^2/s}$$

Example 1 (continued)

(c) How much heat is generated by friction in those six seconds?

The heat gained equals the mechanical energy lost.

$$\Delta KE = \frac{1}{2} I\omega^2 - \frac{1}{2} I\omega_0^2 = \frac{1}{2} I(\omega^2 - \omega_0^2)$$

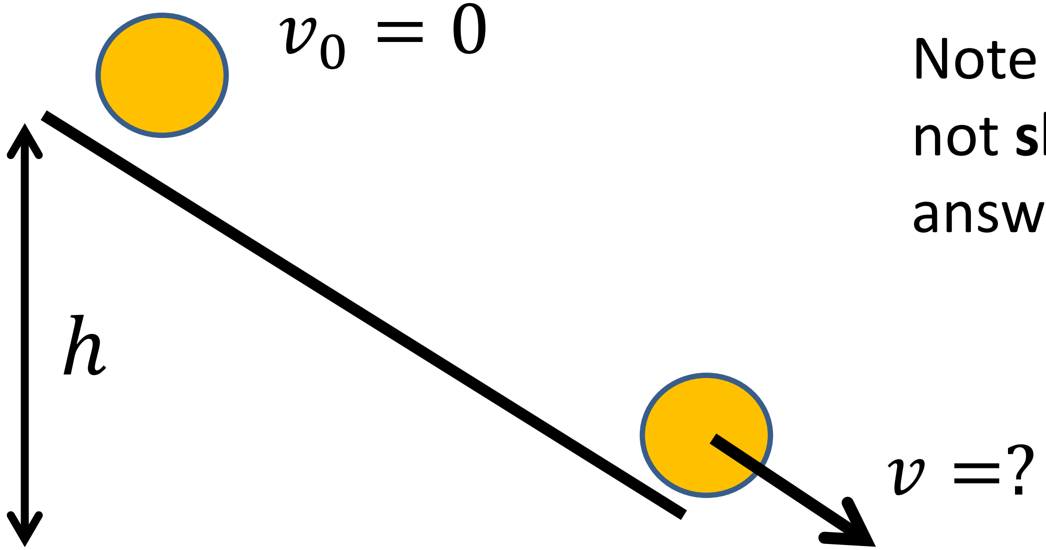
$$= \frac{1}{2}(.045)[\,(82)^2 - (188.5)^2\,]$$

$$= \;-848.2\,J$$

Heat generated: $\boxed{+\,848.2\,J}$

Example 2

(a) A solid sphere of mass m=.1 kg and radius r=.05 m is rolling down a ramp. It starts from rest at a height h=10 m. What is its final speed at the bottom of the ramp?



Note that sphere is **rolling**, not **sliding**. This affects the answer.

Example 2 (continued)

- Initial energy: $U_0 = mgh\,, \quad K_0 = 0$
- Final energy: $U = 0,\ K = \frac{1}{2}\mathrm{m}v^2 + \frac{1}{2}\mathrm{I}\omega^2$

The final K can be simplified:

$$K = \frac{1}{2}\mathrm{m}v^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\left(\frac{v}{r}\right)^2 = \left(\frac{1}{2} + \frac{1}{5}\right)\mathrm{m}v^2 = \frac{7}{10}\mathrm{m}v^2$$

Energy conservation:

$$U_0 + K_0 = U + K \quad \rightarrow \quad mgh = \frac{7}{10}\mathrm{m}v^2$$

$$v = \sqrt{\frac{10}{7}gh} = \sqrt{\frac{10}{7}(9.8)(10)} = \boxed{11.83\ m/s}$$

Example 2 (continued)

(b) What if the object was a cylinder instead of a sphere?

Now, $I = \frac{1}{2}mr^2.$

So: $\frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)^2 = \frac{1}{4}\mathrm{m}v^2.$

Energy conservation becomes:

$$mgh = \frac{1}{2}\mathrm{m}v^2 + \frac{1}{4}\mathrm{m}v^2 = \frac{3}{4}\mathrm{m}v^2$$

$$v = \sqrt{\frac{4}{3}gh} = \sqrt{\frac{4}{3}(9.8)(10)} = \boxed{11.43\ m/s}$$

This is lower than for the sphere: more of the initial energy goes into rotational motion, so there is less for translational speed.

Example 2 (continued)

(b) What if the object was sliding instead of rolling?

Now energy conservation is:

$$mgh = \frac{1}{2}\mathrm{m}v^2$$

$$v = \sqrt{2gh} = \sqrt{2\,(9.8)(10)} = \boxed{14\,m/s^2}$$

Now, **none** of the initial energy goes into rotational motion, so there is more for translational speed. It moves faster than rolling case.

- With rolling, speed depends on shape.
- Without rolling, the speed is larger and independent of shape.

Example 3

- A turntable of mass M and radius R initially rotates at angular speed ω_0. A lump of clay of same mass drops at radius $\frac{R}{2}$. What is the final ω?

$$L_0 = I_0\, \omega_0 = \frac{1}{2} M R^2\, \omega_0$$

$$L = I\, \omega = \frac{1}{2} M R^2\, \omega + M \left(\frac{R}{2}\right)^2\, \omega$$

$$= \left(\frac{1}{2} + \frac{1}{4}\right) M R^2\, \omega = \frac{3}{4} M R^2 \omega$$

Conservation of angular momentum:

$$L = L_0 \;\rightarrow\; \frac{3}{4} M R^2 \omega = \frac{1}{2} M R^2\, \omega_0 \;\rightarrow\; \boxed{\omega = \frac{2}{3}\omega_0}$$

Example 4

- A star of mass $M = 2 \times 10^{30} kg$ and radius $r_0 = 7 \times 10^5 km$ has a rotational period of 30 days.
- At the end of its life-cycle, it runs out of fuel, and as burning stops it drops in temperature. The outward pressure therefore drops (ideal gas law), causing the star to collapse to a smaller radius.
- At the mass above, the star will collapse to a white dwarf of $r = 7 \times 10^4 km$.
- How fast is it spinning after collapse?

Example 4 (continued)

- Initially: $M = 2 \times 10^{30} kg \qquad r_0 = 7 \times 10^8\ m$

$$T_0 = 30\ days = 2.59 \times 10^6\ s$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} = 2.4 \times 10^{-6}\ rad/s$$

- After collapse: $M = 2 \times 10^{30} kg \qquad r = 1 \times 10^7\ m$
- For a sphere $I = \frac{2}{5} M r^2$, so:

$$I_0 = \frac{2}{5}(2 \times 10^{30})(7 \times 10^8)^2 = 3.92 \times 10^{47} kg\ m^2$$

$$I = \frac{2}{5}(2 \times 10^{30})(1 \times 10^7)^2 = 8 \times 10^{43} kg\ m^2$$

Example 4 (continued)

- Conservation of angular momentum:

$$L_0 = L \quad \rightarrow \quad I_0\,\omega_0 = I\omega$$

$$\omega = \frac{I_0\omega_0}{I} = \frac{(3.92 \times 10^{47})(2.4 \times 10^{-6})}{8 \times 10^{43}}$$

$$\omega = \boxed{1.18 \times 10^{-2}\ rad/s}$$

- Frequency and period:

$$f = \frac{\omega}{2\pi} = \frac{1.18 \times 10^{-2}}{2\pi} = \boxed{1.88 \times 10^{-3}\ Hz}$$

$$T = \frac{1}{f} = 532\ s \ = \boxed{8.9\ minutes}$$

CH. 13 - FLUIDS

Fluids

- **Fluid:** any material which takes the shape of its container. It has no fixed shape of its own.
- Most common examples:
  - Liquids
  - Gases
  - Plasmas (ionized gases)
- Many other materials act like fluids under some circumstances. Example:
  - Conductive materials can often be treated as a container filled with an electron gas.

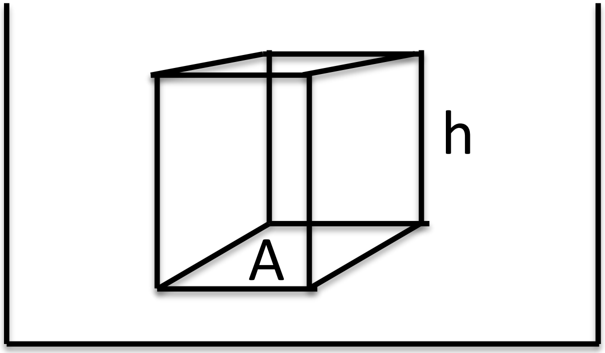
Basic definitions

- **Pressure:** $P = \frac{F}{A}$
  - Units: Pascals, $1\,Pa = 1\,\frac{N}{m^2}$
- **Density:** $\rho = \frac{m}{V}$ (so $m = \rho V$)
  - Units: kg/m^3
- Water: $\rho_{H20} = 1\,\frac{g}{cm^3} = 1000\,\frac{kg}{m^3}$
- Densities of other materials: table in book
- **Specific gravity:** $s.g. = {}^{\rho}/_{\rho_{H20}}$

Example 1

- Given a tank of liquid of density ρ. What is the pressure at depth h below the surface?
- Take an arbitrary area A at depth h.
- Find the weight of fluid above A, then divide by area to find pressure:

$$P = \frac{F}{A} = \frac{mg}{A} = \frac{\rho g V}{A} = \frac{\rho g A h}{A} = \rho g h$$



$$\boxed{P = \rho g h}$$

(Gauge pressure of static fluid)

Static Fluids

Note a few things:

1. Notice that $P = \rho g h$ is only true for **static** fluids.
2. The expression above is potential energy per volume. For **moving** fluids there will also be a term corresponding to kinetic energy per volume (Bernoulli's equation, below)
3. Pressure is a **scalar**, not a vector. The pressure is the same in **all** directions.
4. The pressure found in the last example is the gauge pressure. If we include the air pressure above the surface we get the **absolute pressure**: $P_{gauge} + P_{atm}$

Other units

- In addition to Pascals, other units are often used:
- **Atmospheres:** $1\, atm = 1.013 \times 10^5\, Pa$
- **Torrs** or **mm-Hg**:
  - 1 torr = 133 Pa
  - 1 atm = 760 torr

Note that the ***actual*** atmospheric pressure varies, depending on temperature, altitude, and other factors.

The ***unit*** of atmospheres is roughly the average value of the actual air pressure at sea level.

Example 2

(a) Find the gauge and absolute pressure the bottom of a 2m-deep swimming pool.

$$P_{gauge} = \rho g h = (1000\,\frac{kg}{m^3})(9.8\,\frac{m}{s^2})(2\text{m}) = 19{,}600\ Pa$$

$$P = P_{gauge} + 1\ atm = 19{,}600\ Pa + 1.013 \times 10^5\ Pa$$

$$= \boxed{1.21 \times 10^5\ Pa}$$

(b) If the pool bottom has area 1000 m², find the total force on the pool bottom.

$$F = PA = (1.21 \times 10^5\ Pa)(1000\ m^2) = \boxed{1.21 \times 10^8 N}$$

Example 3

- A mercury barometer, as shown, is used to measure air pressure, P. What is P when the Hg is higher on one side by 760 mm? ($\rho_{Hg} = 1.36 \times 10^4 \frac{kg}{m^3}$)

Let A = area of tube (cross-section).

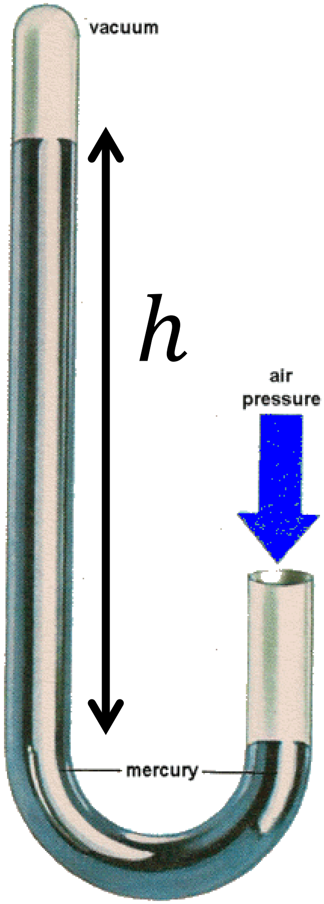
Must have:

$$(weight\ of\ Hg\ on\ left) = (weight\ of\ air\ on\ right)$$

$$\rho_{Hg} ghA = F_{G,air} \rightarrow \rho_{Hg} ghA = P_{air} A$$

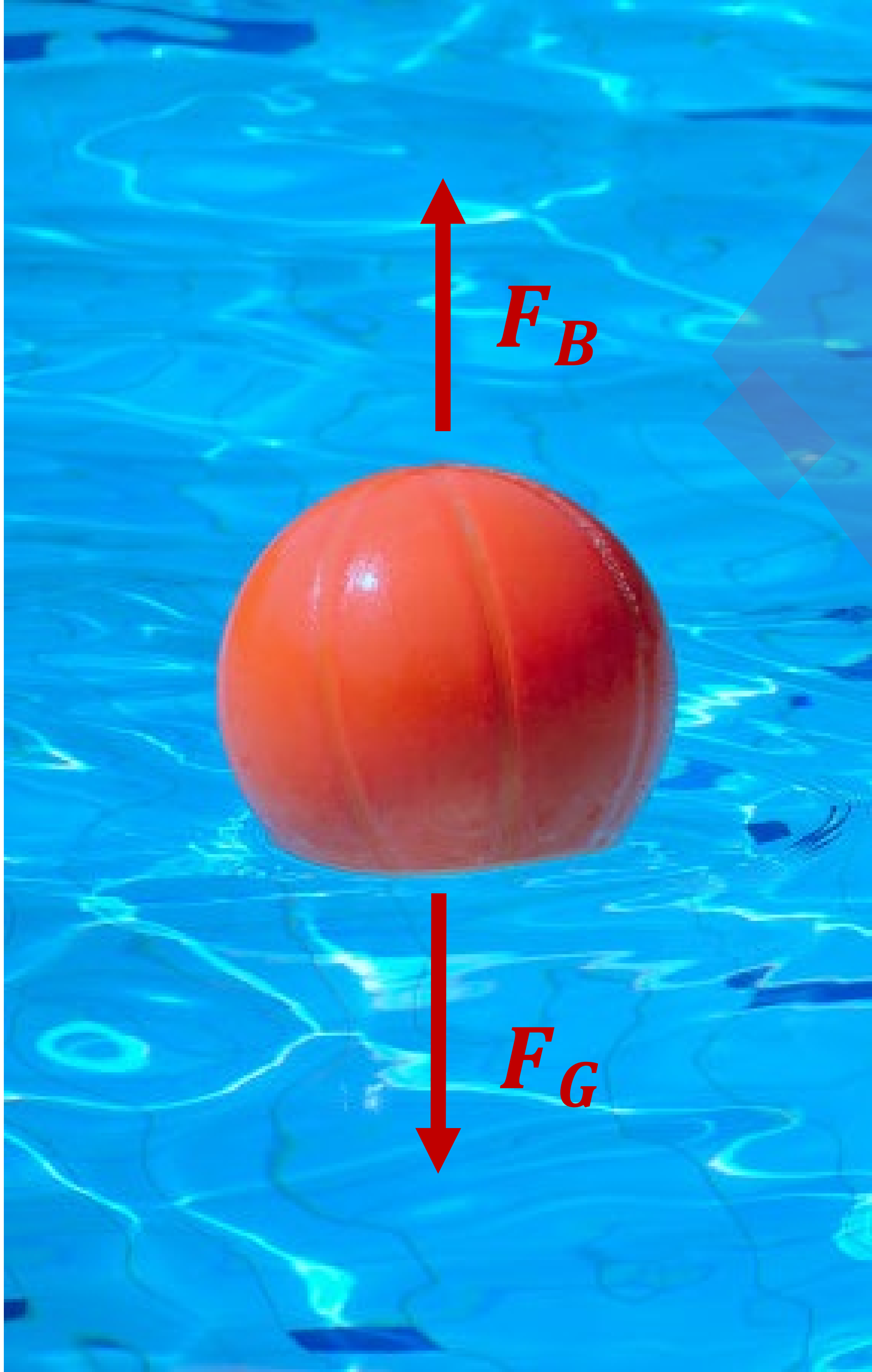
$$P_{air} = \rho_{Hg} gh = \left(1.36 \times 10^4 \frac{kg}{m^3}\right)(9.8)(.76\text{m})$$

$$P_{air} = \boxed{1.015 \times 10^5 Pa = 1.00\ Atm}$$



Bouyancy

- Drop a solid object of density ρ_{obj} into a fluid of density ρ_F.
- Then if you measure the apparent weight of the object, it will be smaller than the actual weight.
- This is due to an upward force exerted on the object by the fluid, partially (or completely) canceling the weight.
- This upward force is called a buoyant force, F_B.



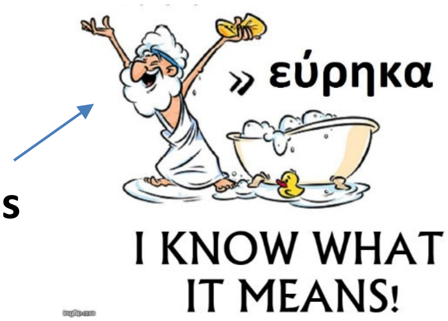
Bouyant Force

- Where does the buoyant force come from?
- Gravity acts on the **water**, as well as the ball.
- In order to pull the ball down, some of the water has to be pushed up to make room for it.
- Similarly, for the water to be pulled down, the ball has to be pushed up out of its way.
- That upward push by the water is the buoyant force.

Archimedes' Principle

- **Archimedes' Principle** - The buoyant force equals the weight of the fluid displaced by the object.
- In equation form: $\boxed{F_B = \rho_F g V_d}$
- Here ρ_F is the density of the fluid, not the object, and V_d is the density of fluid displaced by the object (the portion of the object submerged below the surface)

Archimedes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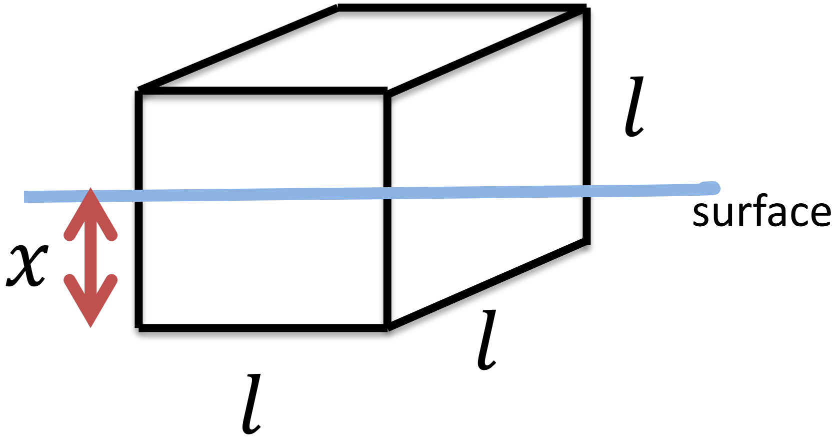
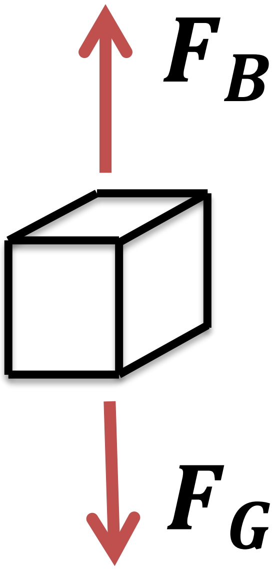
I KNOW WHAT IT MEANS!

Example 4

- An object of density $\rho = 800\,\frac{kg}{m^3}$ is in the shape of a cube of length $0.1\ m$ on each side.
- How much of the block will be submerged when placed in water?


l
surface
x
l
l


$\boldsymbol{F_B}$
$\boldsymbol{F_G}$

Example 4 (continued)

- Mass of object:

$$m = \rho V = \rho l^3 = \left(800\ \frac{kg}{m^3}\right)(.1\ m)^3 = .8\ kg$$

- Mass of water displaced:

$$m_{h20} = \rho_{h20} V_{h20} = \rho_{h20} l^2 x = \left(1000\ \frac{kg}{m^3}\right)(.1\ m)^2 x = 10x$$

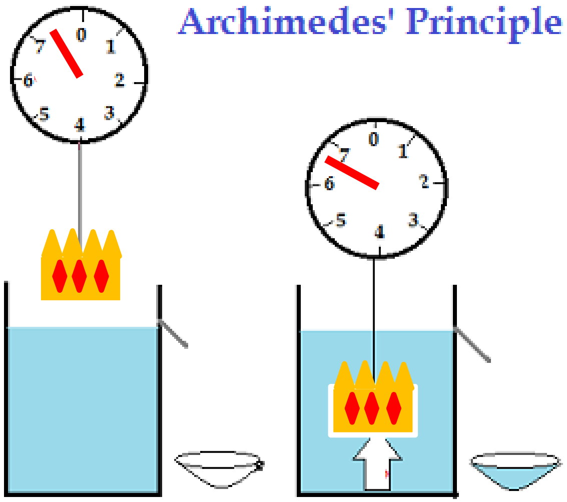
- $\Sigma F = 0 \quad \rightarrow \quad F_G = F_B \quad \rightarrow \quad mg = m_{h20}\mathrm{g}$
- So part of height submerged is:

$$.8 = 10x \quad \rightarrow \quad x = \frac{.8}{10} = \boxed{.08\ \mathrm{m}}$$

80% submerged, 20% above water. Notice that the fraction below the water equals the specific gravity of the object.

Example 5

- Princess Di wants to know if her crown is really made of gold. She weighs the crown and find that $F_G = 7.84\ N$.
- She then weighs it again while submerged in water, finding an apparent weight of 6.86 N.


Archimedes' Principle
0
1
2
3
4
5
6
7
0
1
2
3
4
5
6
7

Is the crown made of gold?

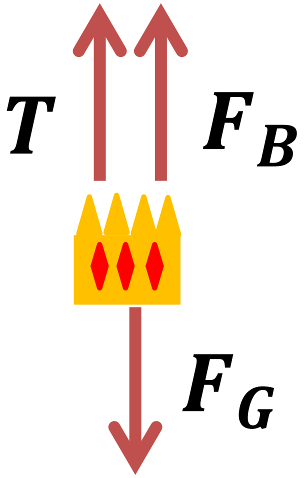
Given: $\rho_{gold} = 1.93 \times 10^4 \frac{kg}{m^3}$

Note that the apparent weight is just the tension

Example 5 (continued)

In water: $\Sigma F = ma \rightarrow T + F_B - F_G = 0$

$$F_B = F_G - T = 7.84 - 6.86 = .98\text{ N}$$



But: $F_B = \rho_{h2o} g V_d$

So: $V_d = \dfrac{F_B}{\rho_{h2o} g} = \dfrac{.98}{(9.8)(1000)} = 10^{-4} m^3$

Since $V_d = V_{crown}$:

$$m = \frac{F_G}{g} = \frac{7.84}{9.8} = .8\, kg$$

$$\rho_{crown} = \frac{m}{V_d} = \frac{.8}{10^{-4}} = 8000\ \frac{kg}{m^3} \neq \rho_{gold}$$

Not Gold!

Example 6

(a) An empty balloon has a mass of $m_b = .012kg$. It is filled with helium at density $\rho_{He} = .181\,\frac{kg}{m^3}$. The balloon then has radius $r = 0.5\ m$. If the balloon is fastened to a vertical string, what is the tension in the line? $(\rho_{air} = 1.29\,\frac{kg}{m^3})$

First:

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(0.5)^3 = \ .524\ m^3$$

$$m = m_b + m_{He} = m_b + \rho_{He}V = .012 + (.181)(.524)$$
$$= .107\ \text{kg}$$

Now apply F = ma

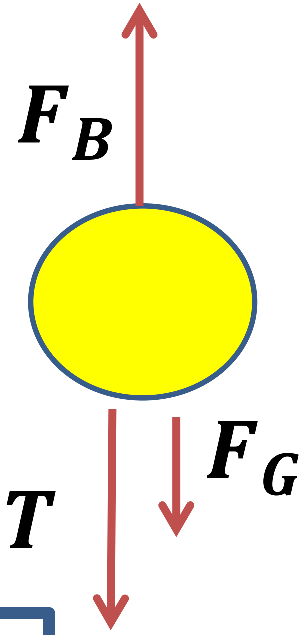
Example 6 (continued)

- The buoyant force is the weight of the air displaced by the balloon:

$$F_B = \rho_{air} g V = (1.29)(9.8)(.524) = 6.624\ N$$

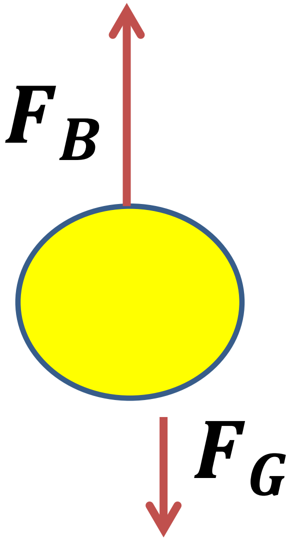
$$\Sigma F_y = m a_y \quad \rightarrow \; F_B - mg - T = 0$$

$$T = \; F_B - mg = 6.624 - (.107)(9.8) = \boxed{5.57\ N}$$



Example 6 (continued)

(b) The string snaps. How fast does the balloon accelerate upward?



Now the tension is gone but a_y is not 0.

$$\Sigma F_y = ma_y \quad \rightarrow \quad F_B - mg = ma_y$$

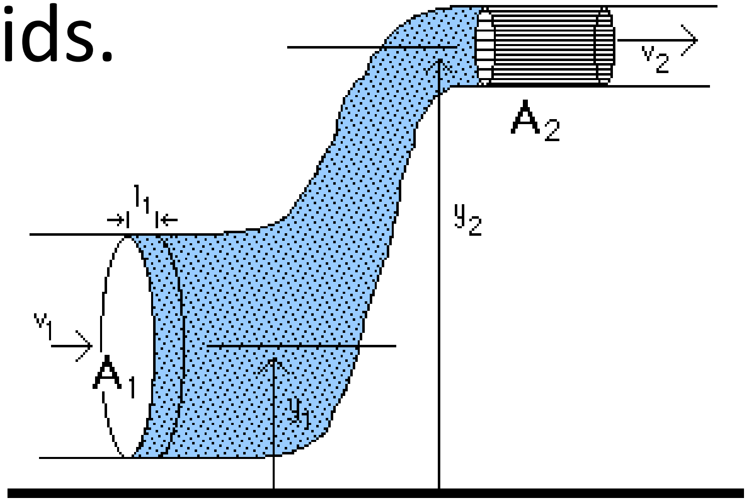
$$a = \frac{F_B - mg}{m} = \frac{6.624 - (.107)(9.8)}{.107}$$

$$= \boxed{+52.1\, m/s^2}$$

upward

Flowing Fluids

- Until now we have looked at fluids at rest. Now we look at moving fluids.



- There are two main facts:

1. **Continuity equation:** $\boxed{A_1 v_1 = A_2 v_2}$
2. **Bernoulli's equation:**

$$\boxed{P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2}$$

Continuity equation

- Both of these equations have a simple meaning.
- Notice that:

$$Av = A\left(\frac{\Delta x}{\Delta t}\right) = \frac{A\Delta x}{\Delta t} = \frac{\Delta V}{\Delta t} = volume\ per\ time$$

- So the continuity equation is simply telling us that the volume per time of fluid in equals the volume per time out.
- Volume is conserved: no fluid disappears inside the pipe.

Note that A is the area of the cross-section, not the surface area.

Bernoulli's equation

- Bernoulli's equation can be written as:

$$P + \frac{1}{2}\rho v^2 + \rho g y = \text{constant}$$

- Note that the second and third terms are kinetic and potential energy per volume.
- The first term is work per volume:

$$P = \frac{F}{A} = \frac{F\Delta x}{A\Delta x} = \frac{W}{\Delta V}$$

- So Bernoulli is just conservation of energy (or work-energy theorem) per volume.

Example 7

At rest, the heart pumps blood into the aorta at a rate of $6 \times 10^{-5}\,\frac{m^3}{s}$. The cross section of the aorta is $8 \times 10^{-5} m^2$.

(a) What is the speed of the blood through the aorta?

$$volume\ per\ time = A_1 v_1$$

So:

$$v_1 = \frac{vol/time}{A_1} = \frac{6 \times 10^{-5}\,\frac{m^3}{s}}{8 \times 10^{-5} m^2} = \boxed{0.75\ m/s}$$

Example 7 (continued)

(b) The blood spreads into capillaries. There are 5×10^6 of them, of average radius 4×10^{-6}m.

Find the average speed v_2 of the blood through the capillaries.

Total area:

$$A_2 = (\#\ of\ caps.)(area\ per\ cap.)$$

$$= (5 \times 10^6)(\pi r^2) \ = \ (5 \times 10^6)\pi(4 \times 10^{-6})^2$$

$$= 2.5 \times 10^{-4}\ m^2$$

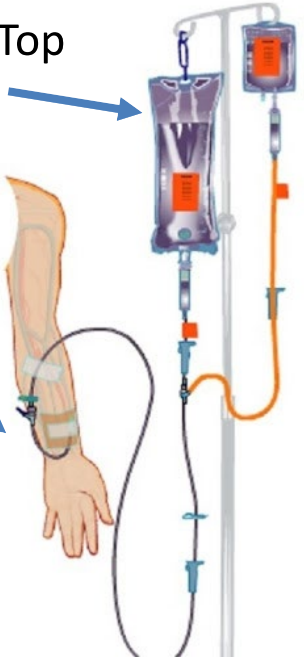
$$A_1 v_1 = A_2 v_2 \rightarrow v_2 = \frac{A_1 v_1}{A_2} = \left(\frac{(8\times10^{-5})(.75)}{2.5\times10^{-4}}\right) = \boxed{.24\ m/s}$$

Example 8

Intravenous fluid:

- Assume density is same as for water.
- At top of fluid: $P_1 = 1\,atm$
- At needle: $P_2 = 1.2\,atm$ (blood pressure)
- So $P_2 - P_1 = .2\,atm \approx 2 \times 10^3 Pa$
- We want the flow rate from the needle into the vein to be:

$$\frac{\Delta V}{\Delta t} = 6.6 \times 10^{-8}\,\frac{m^3}{s}$$



Point 1: Top of fluid

Point 2: needle

Example 8 (continued)

(a) At what speed does the fluid leave the needle and enter the vein? (v_2 = ?) Assume the cross section of the needle is $A_2 = 4 \times 10^{-7}\ m^2$.

We know that: $volume\ per\ time = A_2 v_2$

So: $$v_2 = \frac{Vol.per\ time}{A_2} = \frac{6.6 \times 10^{-8}\,\frac{m^3}{s}}{4 \times 10^{-7} m^2}$$

$$= \boxed{.165\ {}^{m}/{}_{s}}$$

Example 8 (continued)

(b) To what height above the arm must the IV bag be raised, $h = y_1 - y_2 = ?$ (Assume $v_1 \ll v_2$.)

$$P_1 + \cancel{\frac{1}{2}\rho v_1^2} + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

v_1 small

$h = y_1 - y_2$

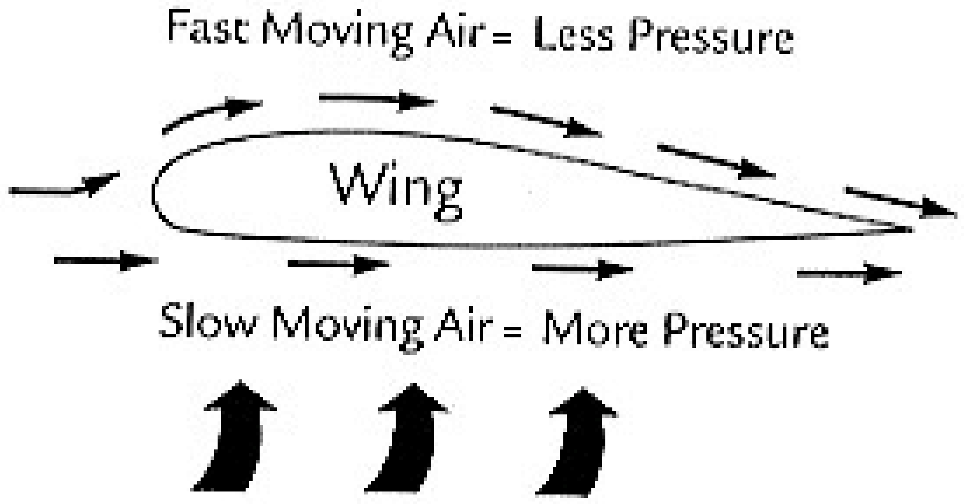
$$0 = (P_2 - P_1) + \frac{1}{2}\rho v_2^2 + \rho g(y_2 - y_1) = (P_2 - P_1) + \frac{1}{2}\rho v_2^2 - \rho g h$$

$$\rho g h = (P_2 - P_1) + \frac{1}{2}\rho v_2^2$$

$$h = \frac{(P_2 - P_1) + \frac{1}{2}\rho v_2^2}{\rho g} = \frac{2 \times 10^3 + \frac{1}{2}(1000)(.165)^2}{(1000)(9.8)} = \boxed{0.20\ m}$$

Bernoulli and Flight

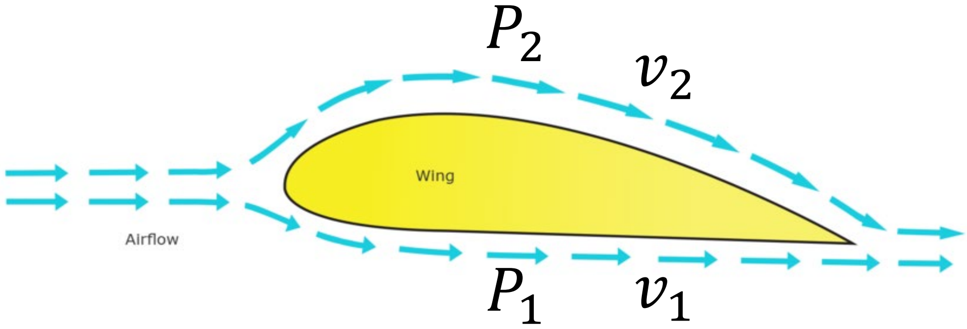
- Airplane wings are designed so air has to flow faster over the wing than under it.
- Bernoulli then implies that the pressure above the wing is smaller than below.
- The pressure difference cause an upward lift force on the wing.


Fast Moving Air = Less Pressure
Wing
Slow Moving Air = More Pressure

Example 9

Given:

$m = 3 \times 10^5\ kg$ (airplane mass)
$v_1 = 30\ m/s$ (under wing)
$v_2 = 40\ m/s$ (over wing)
$\rho = 1.29\ kg/m^3$ (air density)
$A = 500\ m^2$ (area of wing)



- Will the airplane be able to take off?

$$P_1 + \frac{1}{2}\rho v_1^2 + \cancel{\rho g} y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \cancel{\rho g} y_2 \qquad (since\ y_1 \approx y_2)$$

$$P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2)$$

$$F_{lift} = (P_1 - P_2)A = \frac{1}{2}\rho(v_2^2 - v_1^2)A = \frac{1}{2}(1.29)\left((40)^2 - (30)^2\right)(500)$$

$$= 2.26 \times 10^6 N$$

$$F_G = mg = (3 \times 10^5)(9.8) = 2.94 \times 10^6\ N$$

Can't fly:
$F_G > F_{lift}$

Hurricanes!

- Hurricane is passing a house.
- Winds just above roof (point 2): $v_2 = 150\,mph = 67\,m/s$
- Air is at rest inside house (point 1, just below roof): $v_1 = 0$
- Thickness of roof is negligible: $y_1 \approx y_2$

Bernoulli says:

Density of air

$$P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) + \cancel{\frac{1}{2}\rho(y_2 - y_1)} = \frac{1}{2}\rho v_2^2 = \frac{1}{2}(1.29)(67)^2$$

$$= 2.9 \times 10^3\ Pa \quad = .4\ lb\ per\ square\ inch$$

Force on roof of area 2000 m^2 : $F = (P_1 - P_2)A = \boxed{5.8 \times 10^6\ N}$

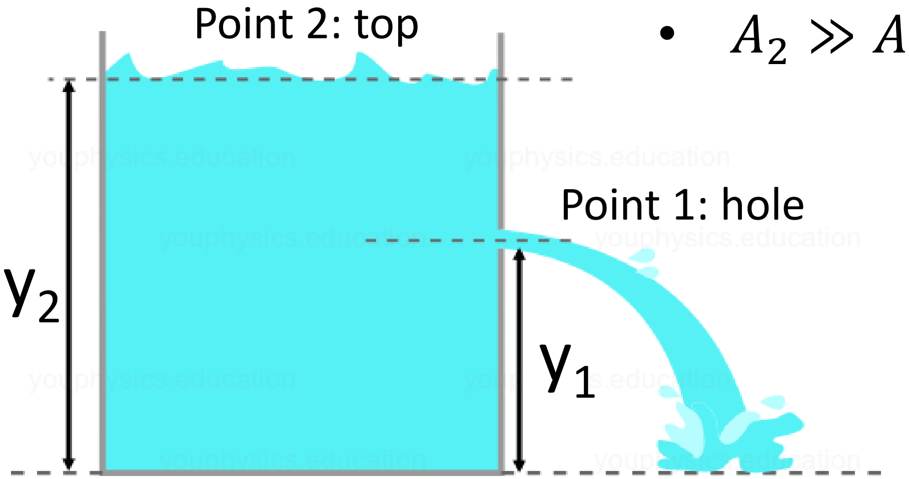
Even faster speeds occur in tornadoes: speeds up to 260 mph=115 m/s can produce forces 4 times larger than this. These forces are more than sufficient to rip the roof off a house.

Toricelli's Law

- A small hole is made in a container of fluid.
- Top of the container is open to atmosphere.
- What is speed of fluid out hole?

Observe:

- Top open to atmosphere → $P_2 = 1\ atm$
- Hole is open to atmosphere → $P_1 = 1\ atm$
- $A_2 \gg A_1 \quad \rightarrow \quad v_2 \ll v_1$ (by continuity eq.)



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Toricelli’s Law

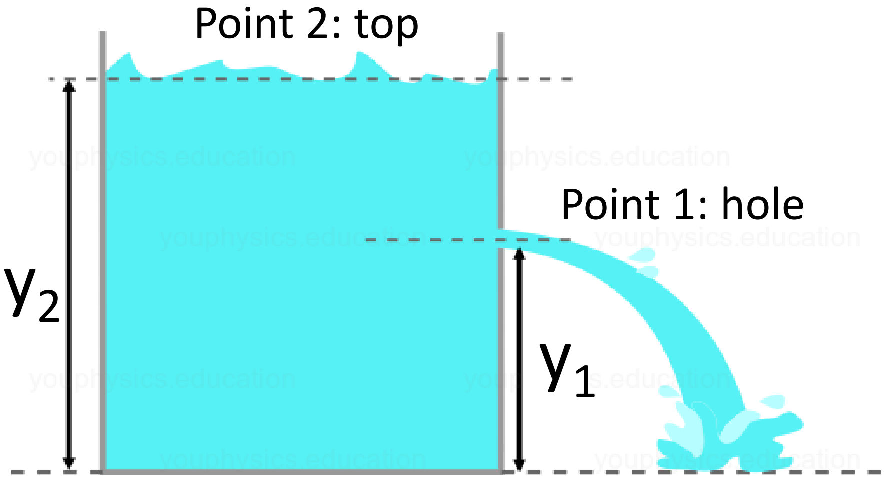
$$\cancel{P_1} + \frac{1}{2}\rho v_1^2 + \rho g y_1 = \cancel{P_2} + \frac{1}{2}\rho \cancel{v_2^2} + \rho g y_2$$

(Since $P_1 = P_2$ and v_2 small)

$$\frac{1}{2}\rho v_1^2 + \rho g y_1 = \rho g y_2$$

$$\frac{1}{2} v_1^2 + g y_1 = g y_2 \quad \rightarrow \quad \boxed{v_1 = \sqrt{2g(y_2 - y_1)}}$$

(Toricelli’s law)



Point 2: top
Point 1: hole
y_2
y_1

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Example

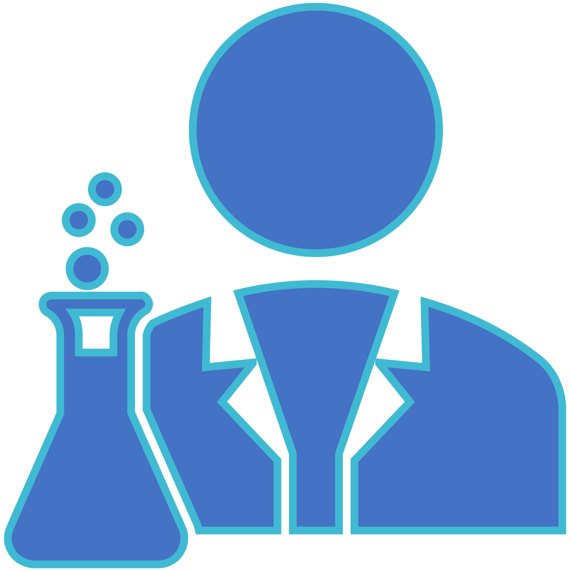
- A water tank filled to a height of .5 m has a small hole of area $A_1 = 1\,cm^2 = 10^{-4}m^2$ near the bottom.
- What volume of water is lost through the leak per second?

Toricelli:

$$v_1 = \sqrt{2g(y_2 - y_1)} = \sqrt{2(9.8)(0.5m)} = 3.13\ \frac{m}{s}$$

Volume per time:

$$\frac{\Delta(vol)}{\Delta t} = \mathrm{Av} = (10^{-4})(3.13) = \boxed{3.13 \times 10^{-4}\,\frac{m^3}{s}}$$



Ch. 17 – Temperature and Ideal Gases

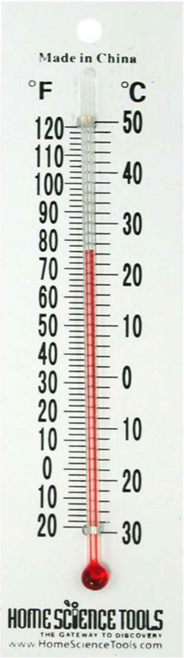
Some questions

- Temperature will be denoted by capital T.
- What is temperature? What does it actually measure?
- What is heat?
- How are temperature and heat related, and what is their connection to mechanical energy?

- Answering these is the goal of this chapter and the next two.

How is temperature measured?

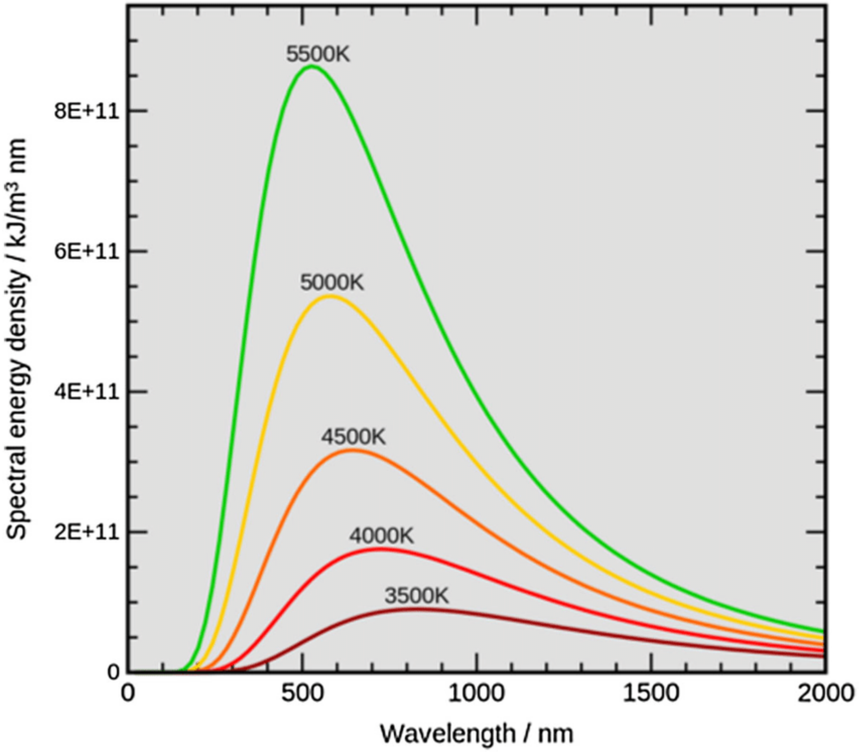
- There are two common ways to measure temperature.
- One way is to measure the expansion of some material. Most materials expand an amount proportional to their change in temperature.


Made in China
°F
°C
120
110
100
90
80
70
60
50
40
30
20
10
0
10
20
50
40
30
20
10
0
10
20
30
HOME SCIENCE TOOLS
THE GATEWAY TO DISCOVERY
www.HomeScienceTools.com

- Common household thermometers use mercury or alcohol in a closed tube. A scale is marked next to the column to measure the volume change

How is temperature measured?

- For more distant objects, or objects that you can't bring a thermometer into contact with there is a second method.
- Measure the output of electromagnetic radiation (light, UV, IR, etc) at different wavelengths. The location of the peak in the curve is used to determine temperature.
- These *blackbody radiation curves* will be discussed next semester.



Units of temperature

- Common units are:
  - Fahrenheit ($^\circ F$), Celsius or centigrade ($^\circ C$), and Kelvin (K).

Boiling point of water	**212°F**	**100°C**	**373 K**
Freezing point of water	**-32°F**	**0°C**	**273 K**
Absolute zero	**-459°F**	**-273°C**	**0 K**

- **Absolute zero** = lowest possible temperature (all molecules at rest)
- Room temperature $\approx 20^\circ C$
- Body temperature $\approx 37^\circ C \approx 98.6^\circ F$
- Microwave background (outer space) $\approx 2.73\ K$

Units of temperature

- Unit conversion:

$$T(°\mathrm{C}) = \frac{5}{9}\,[T(°\mathrm{F}) - 32]$$

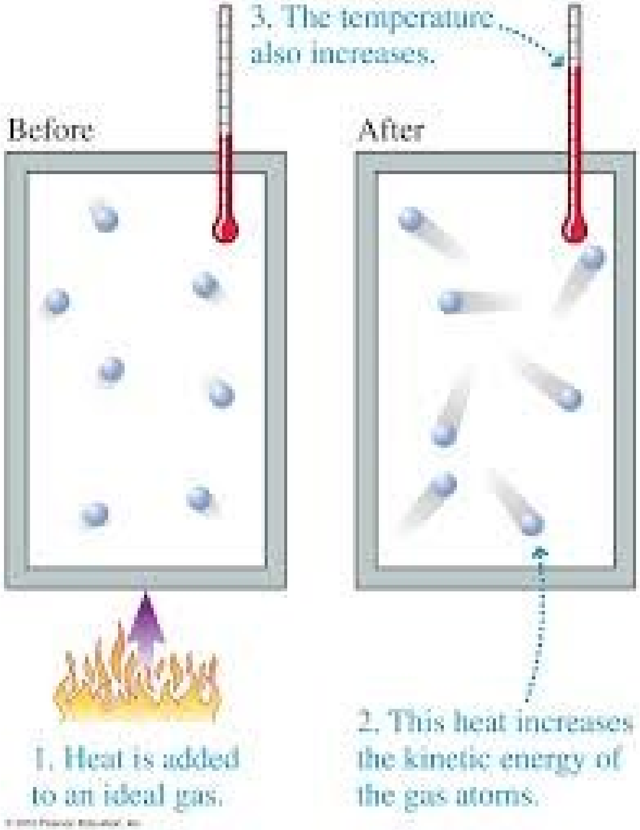
$$T(°\mathrm{F}) = \frac{9}{5}T(°\mathrm{C}) + 32$$

$$T(K) = T(°\mathrm{C}) + 273$$

- **Danger**: Most formulas require temperatures to be in **Kelvins**.
- **Exceptions:** formulas involving temperature ***differences*** can be in *either* Kelvin or Celsius

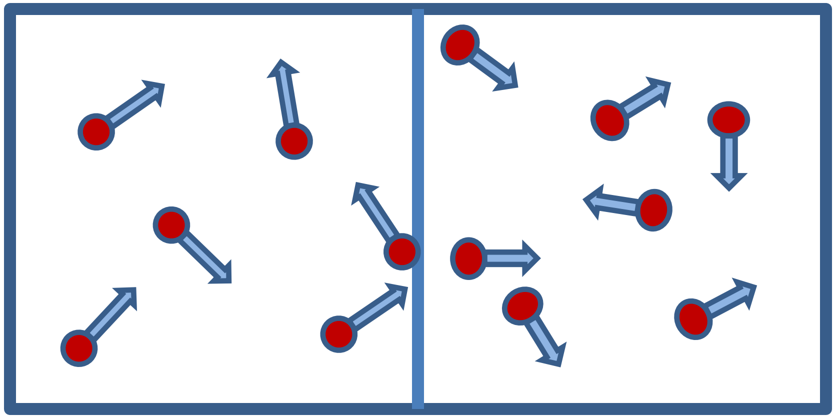
What does temperature measure?

- Roughly, it is a measure of average translational kinetic energy per molecule in a material.
- This statement will be made more precise in the next chapter.
- Notice the word “*translational*” above. Rotational kinetic energy, or energy stored in internal vibrations of a molecule don’t contribute to the temperature.


3. The temperature also increases.
Before
After
1. Heat is added to an ideal gas.
2. This heat increases the kinetic energy of the gas atoms.

Heat and conduction

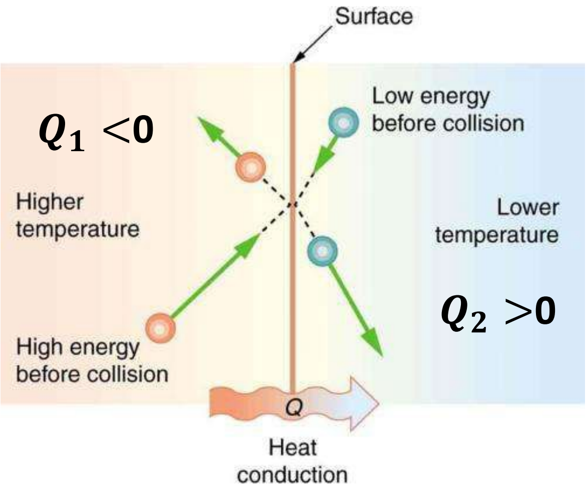
- Imagine two objects at different temperatures in contact. Two ideal gases for example:
- Suppose $T_1 > T_2$.
- Molecules collide elastically with the barrier.
- As a result, kinetic energy is exchanged between the two gases



Object 1 (hot)
T_1
$Q_1 < 0$

Object 2 (cold)
T_2
$Q_2 > 0$

Heat and conduction


Surface
$Q_1 < 0$
Low energy before collision
Higher temperature
Lower temperature
$Q_2 > 0$
High energy before collision
Q
Heat conduction

- Since initially the molecules on the left had more kinetic energy, statistically more energy will go from left to right than will go the other way.
- On average, energy will flow from hot side (left) to cold side (right).
- In other words, the hot side cools and the cold side heats up.
- This continues until both sides are the same temperature (thermal equilibrium).
- This process is called **conduction**.
- The energy transferred is called **heat.**

Heat and conduction

- We now have two types of energy transfer:
  - $\boldsymbol{W}$ = **work** = energy transfer due to forces
  - $\boldsymbol{Q}$ = **heat** = energy transfer due to temperature differences
- Signs of Q:
  - $Q > 0$: object gains energy (the hotter object)
  - $Q < 0$: object loses energy (the cooler object)
- Besides conduction, there are two other ways heat can be exchanged (see book for these):
  - Radiation (electromagnetic waves carry the energy)
  - Convection (mixing of hot and cold fluids)

Ideal gases

- A gas is called an **ideal gas** if:
  - The molecules have zero volume: they are point masses
  - There are no interactions between the molecules except when they elastically collide.
- These are approximations, but for most gases at normal pressures and temperatures, ideal gases are excellent approximations to real gases.
- The approximations break down at very high or very low temperatures. In these circumstances correction terms are added, which give the **van der Waals equation.**
- To study ideal gases, we’ll need some definitions

Standard Temperature and Pressure

- It is convenient to define a standard set of reference conditions.
- Many quantities are defined under these conditions, and then extrapolated to other conditions.
- **Standard temperature and pressure (STP)** is defined as $T = 273\,K = 0°\mathrm{C}\,, \quad P = 1\,atm = 1.013 \times 10^5\,Pa$

Atomic mass units

- The kilogram is a large unit to be using for molecules or other microscopic objects. So we define a smaller unit.
- The **atomic mass unit, u**, is defined by setting the mass of the carbon-12 atom, ^{12}C, exactly equal to 12 u.
- Mass of hydrogen atom: M(H) =1.0078 u
- Masses of proton and neutron approximately equal: $M(p) = M(n) \approx 1u$.
- So, the mass in u of an atom is approximately equal to the number of protons and neutrons in the nucleus.
- Conversion: $1\,u = 1.6605 \times 10^{-27} kg$
- **Warning:** when plugging into formulas. masses must be in kg, not u,

Moles

- **Moles** can be defined several ways.

1. 1 mole of ideal gas is the amount that occupies $22.4 \times 10^{-3} m^3$ (22.4 liters) of volume at STP. (This definition applies only to gases)
2. 1 mole of any material contains **Avagadro's number**, N_A, of molecules: $N_A = 6.02 \times 10^{23}$.
3. 1 mole of material is the amount whose mass in grams equals the mass of one molecule in u.

Examples

- 1 mole of carbon-12 has mass of 12 g.
- 1 oxygen atom has 8 protons and 8 neutrons. So, 1 mole of *atomic* oxygen has mass of 16 g.
- 1 mole of *molecular* oxygen, O_2, has mass of 32 g

- Some notation: $N =$ # of molecules, $n =$# of moles
- Note that: $\boxed{N = nN_A}$

Ideal Gas Law

- Two versions:

$$\boxed{\begin{aligned} PV &= nRT \\ PV &= NkT \end{aligned}}$$

- The constants are:
  - **Ideal Gas Constant:** $R = 8.315\, J/mol \cdot K$
  - **Boltzmann's constant:** $k = 1.38 \times 10^{-23} J/K$

- Note:
  - k and R are related by: $R = N_A k$
  - T **must** be in Kelvins

Special cases of ideal gas law

- Boyle's law: $PV = \text{constant}$, when T is held fixed.
- Charles' law: $\frac{V}{T} = \text{constant}$, when P is held fixed.
- Guy-Lusac's law: $\frac{P}{T} = \text{constant}$, when V is held fixed.

- These were discovered first, but all of them follow directly from the ideal gas law.

Using the ideal gas law

- Important: when using the ideal gas law,
  - T must be in Kelvins
  - P must be in Pascals
  - V must be in m^3
- Conversion for volumes: $1\,L = 10^{-3}\,m^3$
- Be careful not to confuse n and N.

Summary - molar and molecular quantities:

Molecular	**Molar**	**Relation**
N	n	$N = nN_A$
m	M	$M = mN_A$
R	k	$R = kN_A$

Example 1

- A cylinder contains oxygen at $20℃$, initial pressure $P_0 = 15\ atm$, and initial volume $V_0 = 12\ L$.
- The temperature is raised to $\mathrm{T} = 35℃$, while the volume is reduced to $V = 8.5\ L$.
- What is final pressure, P?

- Strategy: both initial and final values must satisfy $PV = nRT$. Rearrange so all constant are on the same side of the equation.
- Then what ever combination of variables is on the other side must also be constant.

Example 1 (continued)

- In the current case, n and R are constant. So $\frac{PV}{T} = nR$ must also be constant: $\frac{P_0 V_0}{T_0} = \frac{PV}{T}$
- Rearrange to find final pressure:

$$\begin{aligned} P &= P_0 \left(\frac{V_0}{V}\right)\left(\frac{T}{T_0}\right) \\ &= (1.5 \times 10^6 Pa)\left(\frac{1.2\times10^{-2} m^3}{8.5\times10^{-3} m^3}\right)\left(\frac{308K}{293K}\right) \\ &= \boxed{2.2 \times 10^7\ Pa} \end{aligned}$$

Note: in this case, we could have left V in liters, since the units cancel from top and bottom anyway. And we could have left P in atmospheres, which would simply give an answer in atm. But the temperature must be in Kelvins.

Example 2

(a) Find the volume of 1 mole of ideal gas at STP. (The result should verify that the first definition of moles is consistent with the second.)

$$PV = nRT \rightarrow \ V = \frac{nRT}{P} = \frac{(1\,mol)(8.315\ \frac{J}{mol \cdot K})(273\ K)}{(1.013 \times 10^5 Pa)}$$

$$= .0224\,m^3 \ = \boxed{\ 22.4\ L\ }$$

Here, P did not appear in a ratio, so it **had to be** in Pa, not atmospheres. Note that the volume automatically comes out in cubic meters. If you want liters you need to convert.

Example 2 (continued)

(b) What would be the volume of 10^{24} molecules of ideal gas at STP?

This time we are given N instead of n. So use the other version of ideal gas law.

$$PV = NkT \rightarrow \quad V = \frac{NkT}{P} = \frac{(10^{24})(1.38 \times 10^{-23}\,\frac{J}{K})(273\ K)}{(1.013 \times 10^{5} Pa)}$$

$$= .0372\ m^3 \ = \boxed{37.2\ L}$$

Example 3

- A welder using a tank of volume $V = 0.1\, m^3$ fills it with O_2 gas at initial pressure $P_0 = 5 \times 10^5 Pa$ and temperature $T_0 = 47°\text{C}$. Later, because of a leak in the tank, the pressure and temperature have dropped to $P = 4 \times 10^5 Pa$ and $T = 27°\text{C}$.
- Find:
  (a) The initial mass of oxygen in the tank.
  (b) The mass lost.

Example 3 (continued)

(a) Given: $P_0 = 5 \times 10^5 Pa,\ T_0 = 47°\mathrm{C} = 320\mathrm{K},$ and $V = 0.1\, m^3$ (Volume of tank stays constant.)

Using $PV = nRT$:

$$n_0 = \frac{P_0 V}{RT_0} = \frac{(5 \times 10^5 Pa)(0.1\, m^3)}{(8.315 \frac{J}{mol} \cdot K\,)(320K)} = \boxed{18.8\, moles}$$

$$initial\ mass = (\#\ moles)(mass\ per\ mole)$$

$$= (18.8\, moles)\left(32 \frac{g}{mol}\right) = 601.6\ \ g$$

$$= \boxed{.6016\ \ kg}$$

Example 3 (continued)

(b) Final values: $P = 4 \times 10^5 Pa,\ T_0 = 27℃ = 300\text{K},$ and $V = 0.1\, m^3$.

$$n = \frac{PV}{RT} = \frac{(4 \times 10^5 Pa)(0.1\, m^3)}{(8.315 \frac{J}{mol} \cdot K\,)(300K)} = \ 16.0\, moles$$

Number of moles lost:

$$\Delta n = n - n_0 = 16 - 18.8 = -2.8\, moles$$

Mass lost:

$$\Delta m = (-2.8\, moles)\left(32\ \frac{g}{mol}\right) = \boxed{-89.6\ g}$$

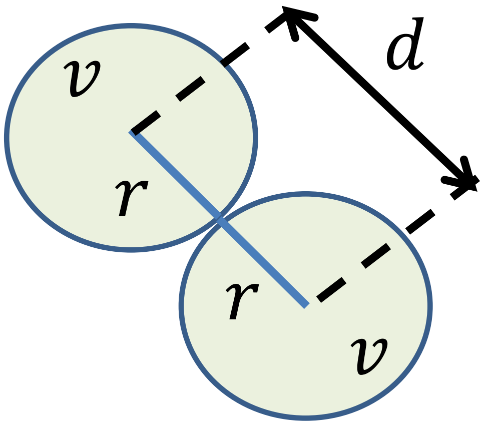
Example 4

- Suppose a container of volume V holds N molecules of gas at STP. Estimate the average distance between the gas molecules.

- Think of each molecule taking up a spherical volume v. Clearly, we must have $v = \frac{V}{N}$.
- But we can define a radius for each sphere: $v = \frac{4}{3}\pi r^3$.
- Using these two expressions, solve for r:

$$r = \left(\frac{3v}{4\pi}\right)^{1/3} = \left(\frac{3V}{4\pi N}\right)^{1/3}$$

Example 4 (continued)



- We have $r = \left(\frac{3v}{4\pi}\right)^{1/3} = \left(\frac{3V}{4\pi N}\right)^{1/3}$.
- But by the ideal gas law, $\frac{V}{N} = \frac{kT}{P}$.
- So:

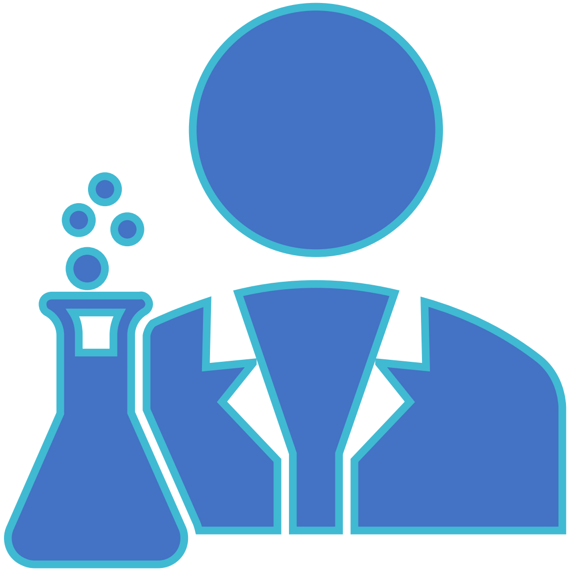
$$r = \left(\frac{3kT}{4\pi P}\right)^{1/3} = \left(\frac{3(1.38 \times 10^{-23})(273)}{4\pi(1.0 \times 10^{5})}\right)^{1/3} = 2.1 \times 10^{-9} m$$

- The average distance between molecules is just twice this:

$$d_{ave} = 2r = 4.2 \times 10^{-9} m = \boxed{4.2\ nm}$$

Next chapter

- In this chapter: simple applications of ideal gas law.
- Next chapter: we'll see where the gas law comes from using a simple mechanical argument.
- Using similar arguments and a bit of statistics, we'll relate temperature to mechanical quantities like kinetic energy and speed.
- This is the beginnings of **statistical mechanics**: the application of statistical methods to find the average mechanical behavior of large numbers of molecules.
- **Thermodynamics** is the study of those large-scale mechanical averages.



Ch. 17 – Temperature and Ideal Gases

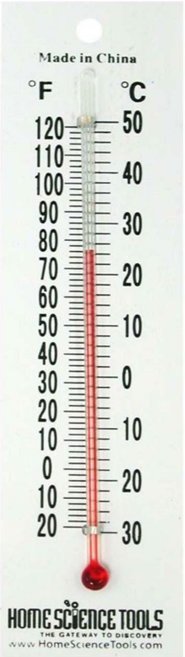
Some questions

- Temperature will be denoted by capital T.
- What is temperature? What does it actually measure?
- What is heat?
- How are temperature and heat related, and what is their connection to mechanical energy?

- Answering these is the goal of this chapter and the next two.

How is temperature measured?

- There are two common ways to measure temperature.
- One way is to measure the expansion of some material. Most materials expand an amount proportional to their change in temperature.


Made in China
°F
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30
HOME SCIENCE TOOLS
THE GATEWAY TO DISCOVERY
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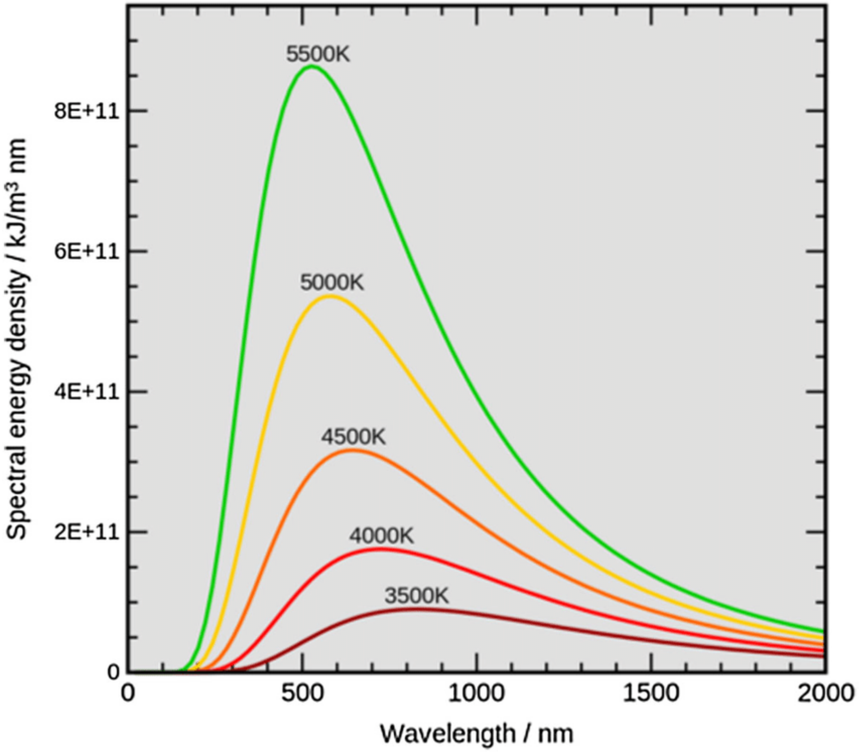
- Common household thermometers use mercury or alcohol in a closed tube. A scale is marked next to the column to measure the volume change

How is temperature measured?

- For more distant objects, or objects that you can’t bring a thermometer into contact with there is a second method.
- Measure the output of electromagnetic radiation (light, UV, IR, etc) at different wavelengths. The location of the peak in the curve is used to determine temperature.
- These *blackbody radiation curves* will be discussed next semester.



Units of temperature

- Common units are:
    - Fahrenheit ($°F$), Celsius or centigrade ($°C$), and Kelvin (K).

Boiling point of water	**212$°F$**	**100$°C$**	**373 K**
Freezing point of water	**-32$°F$**	**0$°C$**	**273 K**
Absolute zero	**-459$°F$**	**-273$°C$**	**0 K**

- **Absolute zero** = lowest possible temperature (all molecules at rest)
- Room temperature $\approx 20°C$
- Body temperature $\approx 37°C \approx 98.6°F$
- Microwave background (outer space) $\approx 2.73\ K$

Units of temperature

- Unit conversion:

$$T(°\mathrm{C}) = \frac{5}{9}\,[T(°\mathrm{F}) - 32]$$

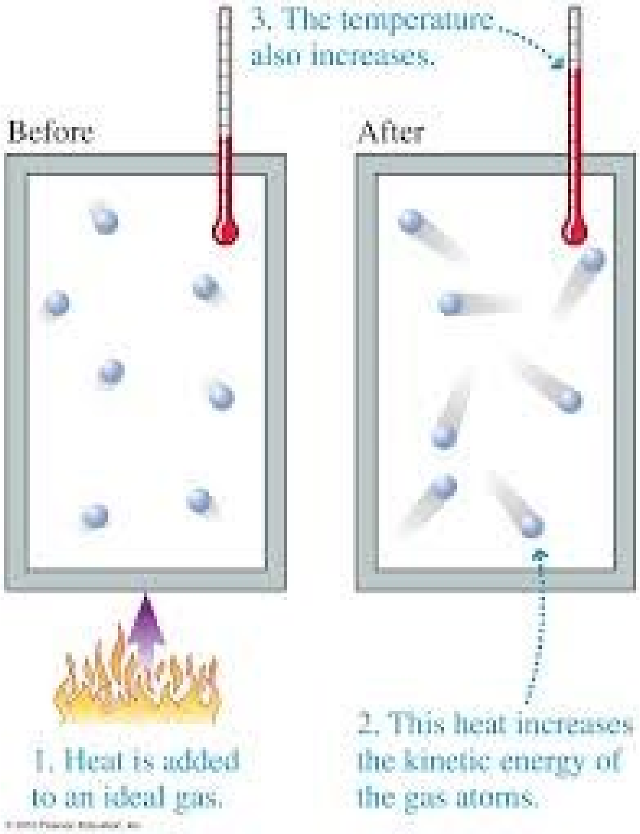
$$T(°\mathrm{F}) = \frac{9}{5}T(°\mathrm{C}) + 32$$

$$T(K) = T(°\mathrm{C}) + 273$$

- **Danger**: Most formulas require temperatures to be in **Kelvins**.
- **Exceptions:** formulas involving temperature ***differences*** can be in *either* Kelvin or Celsius

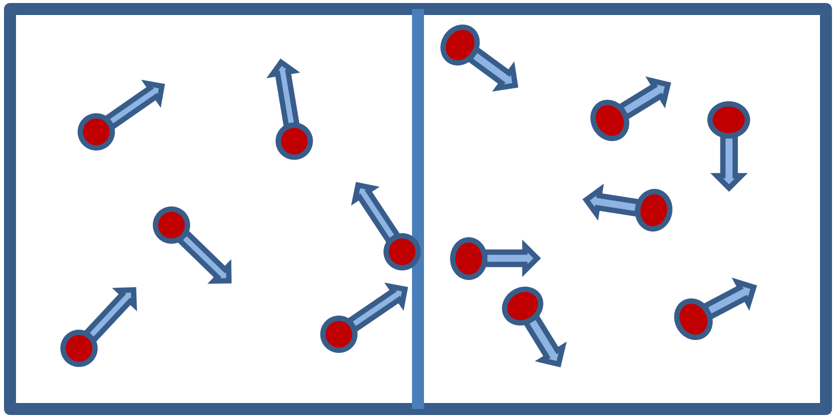
What does temperature measure?

- Roughly, it is a measure of average translational kinetic energy per molecule in a material.
- This statement will be made more precise in the next chapter.
- Notice the word “*translational*” above. Rotational kinetic energy, or energy stored in internal vibrations of a molecule don’t contribute to the temperature.


3. The temperature also increases.
Before
After
1. Heat is added to an ideal gas.
2. This heat increases the kinetic energy of the gas atoms.

Heat and conduction

- Imagine two objects at different temperatures in contact. Two ideal gases for example:
- Suppose $T_1 > T_2$.
- Molecules collide elastically with the barrier.
- As a result, kinetic energy is exchanged between the two gases

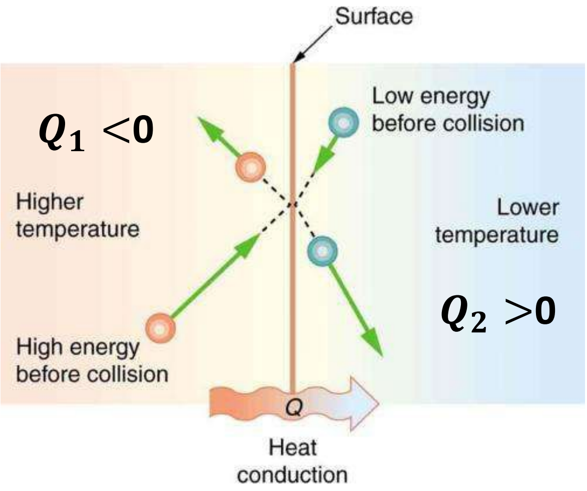


Object 1 (hot)
T_1
$Q_1 < 0$

Object 2 (cold)
T_2
$Q_2 > 0$

Heat and conduction

- Since initially the molecules on the left had more kinetic energy, statistically more energy will go from left to right than will go the other way.
- On average, energy will flow from hot side (left) to cold side (right).



Surface
$Q_1 < 0$
Low energy before collision
Higher temperature
Lower temperature
$Q_2 > 0$
High energy before collision
Q
Heat conduction

- In other words, the hot side cools and the cold side heats up.
- This continues until both sides are the same temperature (thermal equilibrium).
- This process is called **conduction**.
- The energy transferred is called **heat.**

Heat and conduction

- We now have two types of energy transfer:
  - $\boldsymbol{W} =$ **work** = energy transfer due to forces
  - $\boldsymbol{Q} =$ **heat** = energy transfer due to temperature differences
- Signs of Q:
  - $Q > 0$: object gains energy (the hotter object)
  - $Q < 0$: object loses energy (the cooler object)
- Besides conduction, there are two other ways heat can be exchanged (see book for these):
  - Radiation (electromagnetic waves carry the energy)
  - Convection (mixing of hot and cold fluids)

Ideal gases

- A gas is called an **ideal gas** if:
  - The molecules have zero volume: they are point masses
  - There are no interactions between the molecules except when they elastically collide.
- These are approximations, but for most gases at normal pressures and temperatures, ideal gases are excellent approximations to real gases.
- The approximations break down at very high or very low temperatures. In these circumstances correction terms are added, which give the **van der Waals equation.**
- To study ideal gases, we'll need some definitions

Standard Temperature and Pressure

- It is convenient to define a standard set of reference conditions.
- Many quantities are defined under these conditions, and then extrapolated to other conditions.
- **Standard temperature and pressure (STP)** is defined as $T = 273\,K = 0°\mathrm{C}\,,\quad P = 1\,atm = 1.013 \times 10^5\,Pa$

Atomic mass units

- The kilogram is a large unit to be using for molecules or other microscopic objects. So we define a smaller unit.
- The **atomic mass unit, u**, is defined by setting the mass of the carbon-12 atom, ^{12}C, exactly equal to 12 u.
- Mass of hydrogen atom: M(H) =1.0078 u
- Masses of proton and neutron approximately equal: $M(p) = M(n) \approx 1u$.
- So, the mass in u of an atom is approximately equal to the number of protons and neutrons in the nucleus.
- Conversion: $1\,u = 1.6605 \times 10^{-27} kg$
- **Warning:** when plugging into formulas. masses must be in kg, not u,

Moles

- **Moles** can be defined several ways.

1. 1 mole of ideal gas is the amount that occupies $22.4 \times 10^{-3} m^3$ (22.4 liters) of volume at STP. (This definition applies only to gases)
2. 1 mole of any material contains **Avagadro's number**, N_A, of molecules: $N_A = 6.02 \times 10^{23}$.
3. 1 mole of material is the amount whose mass in grams equals the mass of one molecule in u.

Examples

- 1 mole of carbon-12 has mass of 12 g.
- 1 oxygen atom has 8 protons and 8 neutrons. So, 1 mole of *atomic* oxygen has mass of 16 g.
- 1 mole of *molecular* oxygen, O_2, has mass of 32 g

- Some notation: $N =$ # of molecules, $n =$# of moles
- Note that: $\boxed{N = nN_A}$

Ideal Gas Law

- Two versions:

$$\boxed{\begin{aligned} PV &= nRT \\ PV &= NkT \end{aligned}}$$

- The constants are:
  - **Ideal Gas Constant:** $R = 8.315\, J/mol \cdot K$
  - **Boltzmann's constant:** $k = 1.38 \times 10^{-23} J/K$

- Note:
  - k and R are related by: $R = N_A k$
  - T **must** be in Kelvins

Special cases of ideal gas law

- Boyle's law: $PV =$ constant, when T is held fixed.
- Charles' law: $\frac{V}{T} =$ constant, when P is held fixed.
- Guy-Lusac's law: $\frac{P}{T} =$ constant, when V is held fixed.

- These were discovered first, but all of them follow directly from the ideal gas law.

Using the ideal gas law

- Important: when using the ideal gas law,
  - T must be in Kelvins
  - P must be in Pascals
  - V must be in m^3
- Conversion for volumes: $1\,L = 10^{-3}\,m^3$
- Be careful not to confuse n and N.

Summary - molar and molecular quantities:

Molecular	**Molar**	**Relation**
N	n	$N = nN_A$
m	M	$M = mN_A$
R	k	$R = kN_A$

Example 1

- A cylinder contains oxygen at $20^\circ\mathrm{C}$, initial pressure $P_0 = 15\ atm$, and initial volume $V_0 = 12\ L$.
- The temperature is raised to $\mathrm{T} = 35^\circ\mathrm{C}$, while the volume is reduced to $V = 8.5\ L$.
- What is final pressure, P?

- Strategy: both initial and final values must satisfy $PV = nRT$. Rearrange so all constant are on the same side of the equation.
- Then what ever combination of variables is on the other side must also be constant.

Example 1 (continued)

- In the current case, n and R are constant. So $\frac{PV}{T} = nR$ must also be constant: $\frac{P_0 V_0}{T_0} = \frac{PV}{T}$
- Rearrange to find final pressure:

$$P = P_0 \left(\frac{V_0}{V}\right)\left(\frac{T}{T_0}\right)$$

$$= (1.5 \times 10^6 Pa)\left(\frac{1.2 \times 10^{-2} m^3}{8.5 \times 10^{-3} m^3}\right)\left(\frac{308K}{293K}\right)$$

$$= \boxed{2.2 \times 10^7\ Pa}$$

Note: in this case, we could have left V in liters, since the units cancel from top and bottom anyway. And we could have left P in atmospheres, which would simply give an answer in atm. But the temperature must be in Kelvins.

Example 2

(a) Find the volume of 1 mole of ideal gas at STP. (The result should verify that the first definition of moles is consistent with the second.)

$$PV = nRT \rightarrow \quad V = \frac{nRT}{P} = \frac{(1\,mol)(8.315\,\frac{J}{mol \cdot K})(273\,K)}{(1.013 \times 10^5 Pa)}$$

$$= .0224\,m^3 \quad = \boxed{22.4\,L}$$

Here, P did not appear in a ratio, so it **had to be** in Pa, not atmospheres. Note that the volume automatically comes out in cubic meters. If you want liters you need to convert.

Example 2 (continued)

(b) What would be the volume of 10^{24} molecules of ideal gas at STP?

This time we are given N instead of n. So use the other version of ideal gas law.

$$PV = NkT \rightarrow \quad V = \frac{NkT}{P} = \frac{(10^{24})(1.38 \times 10^{-23}\,\frac{J}{K})(273\ K)}{(1.013 \times 10^{5} Pa)}$$

$$= .0372\ m^3 \quad = \boxed{37.2\ L}$$

Example 3

- A welder using a tank of volume $V = 0.1\, m^3$ fills it with O_2 gas at initial pressure $P_0 = 5 \times 10^5 Pa$ and temperature $T_0 = 47$°C. Later, because of a leak in the tank, the pressure and temperature have dropped to $P = 4 \times 10^5 Pa$ and $T = 27$°C.
- Find:
  - (a) The initial mass of oxygen in the tank.
  - (b) The mass lost.

Example 3 (continued)

(a) Given: $P_0 = 5 \times 10^5 Pa,\ T_0 = 47°\text{C} = 320\text{K},$ and $V = 0.1\, m^3$ (Volume of tank stays constant.)

Using $PV = nRT$:

$$n_0 = \frac{P_0 V}{RT_0} = \frac{(5 \times 10^5 Pa)(0.1\, m^3)}{(8.315 \frac{J}{mol} \cdot K)(320K)} = \boxed{18.8\, moles}$$

$$initial\ mass = (\#\ moles)(mass\ per\ mole)$$

$$= (18.8\, moles)\left(32 \frac{g}{mol}\right) = 601.6\ \ g$$

$$= \boxed{.6016\ \ kg}$$

Example 3 (continued)

(b) Final values: $P = 4 \times 10^5 Pa,\ T_0 = 27°\text{C} = 300\text{K},$ and $V = 0.1\, m^3$.

$$n = \frac{PV}{RT} = \frac{(4 \times 10^5 Pa)(0.1\, m^3)}{(8.315 \frac{J}{mol} \cdot K\,)(300K)} = \ 16.0\, moles$$

Number of moles lost:

$$\Delta n = n - n_0 = 16 - 18.8 = -2.8\, moles$$

Mass lost:

$$\Delta m = (-2.8\, moles)\left(32\ \frac{g}{mol}\right) = \ \boxed{-89.6\ g}$$

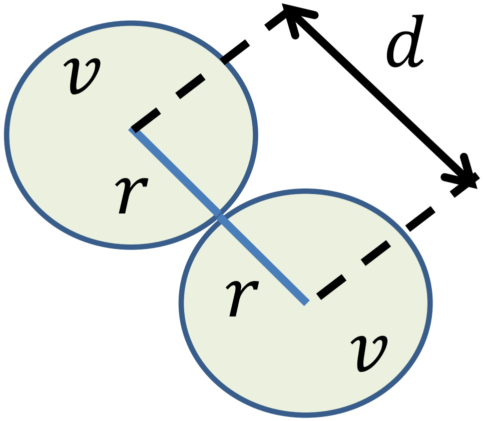
Example 4

- Suppose a container of volume V holds N molecules of gas at STP. Estimate the average distance between the gas molecules.

- Think of each molecule taking up a spherical volume v. Clearly, we must have $v = \frac{V}{N}$.
- But we can define a radius for each sphere: $v = \frac{4}{3}\pi r^3$.
- Using these two expressions, solve for r:

$$r = \left(\frac{3v}{4\pi}\right)^{1/3} = \left(\frac{3V}{4\pi N}\right)^{1/3}$$

Example 4 (continued)



- We have $r = \left(\frac{3v}{4\pi}\right)^{1/3} = \left(\frac{3V}{4\pi N}\right)^{1/3}$.
- But by the ideal gas law, $\frac{V}{N} = \frac{kT}{P}$.
- So:

$$r = \left(\frac{3kT}{4\pi P}\right)^{1/3} = \left(\frac{3(1.38 \times 10^{-23})(273)}{4\pi(1.0 \times 10^{5})}\right)^{1/3} = 2.1 \times 10^{-9} m$$

- The average distance between molecules is just twice this:

$$d_{ave} = 2r = 4.2 \times 10^{-9} m = \boxed{4.2\ nm}$$

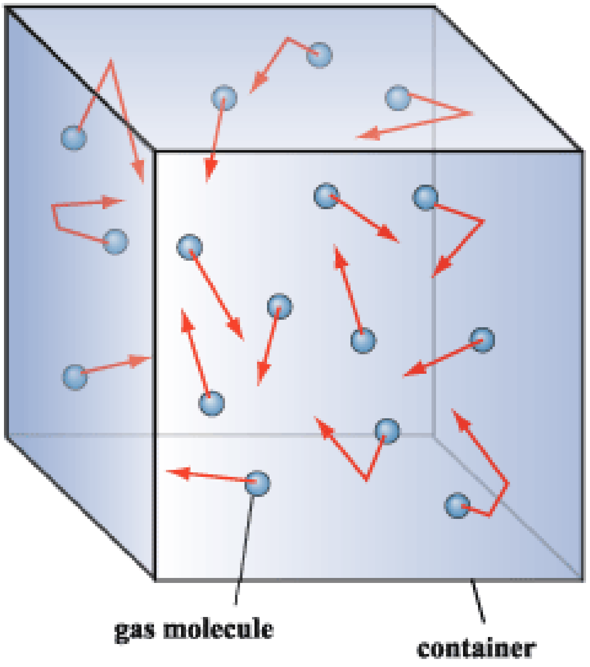
Next chapter

- In this chapter: simple applications of ideal gas law.
- Next chapter: we’ll see where the gas law comes from using a simple mechanical argument.
- Using similar arguments and a bit of statistics, we’ll relate temperature to mechanical quantities like kinetic energy and speed.
- This is the beginnings of **statistical mechanics**: the application of statistical methods to find the average mechanical behavior of large numbers of molecules.
- **Thermodynamics** is the study of those large-scale mechanical averages.

CH. 18 – KINETIC THEORY

Kinetic Theory

- Kinetic theory provides a microscopic picture of the ideal gas law and clarifies the relation of temperature to other quantities.



- Consider an ideal gas in a container. The gas molecules bounce around at random, scattering **elastically** off each other and off the walls.
- For ideal gas, neglect
  - the volume of each molecule
  - the electrostatic interactions between the molecules. (They only interact when they collide)

Pressure

- **Pressure** is due to the collisions of the molecules with the container:

$$P = \frac{F}{A} = \frac{\Delta p}{\Delta t} = p\frac{\Delta n}{\Delta t}$$

- Here, is p the momentum of a single particle, and $\frac{\Delta n}{\Delta t}$ is the number of collisions per time.

- Note that decreasing the volume means that collisions occur more often, in agreement with **Boyle's law**:

$$P \sim \frac{1}{V}\ , \ \text{at fixed } T$$

Temperature

- Temperature is a measure of average kinetic energy per particle. The hotter the material, the faster the molecules bounce around.
- If the particles are moving faster, they should collide more often (larger $\frac{\Delta n}{\Delta t}$), which (according to the last slide means) larger P.
- This gives us **Guy-Lussac's** law:

$$P \sim T \ \ \text{(at fixed } V\text{)}$$

Ideal Gas Law

- We have: $P \sim T$ (at fixed V)

  $P \sim \frac{1}{V}$, at fixed T
- Combining these, $P \sim \frac{T}{V}$
- Clearly, pressure should also be proportional to n, so: $P \sim \frac{nT}{V}$
- If we call the proportionality constant R, then:

$$P = R\frac{nT}{V} \quad \rightarrow \quad \boxed{PV = nRT}$$

Average Kinetic Energy

- The temperature is a measure of the energy.
- To be more precise:

$$\bar{K} = \frac{3}{2}kT \qquad \text{(per mole)}$$

$$\bar{K} = \frac{3}{2}RT \qquad \text{(per molecule)}$$

- Be careful: K here is the translational kinetic energy only (no rotations or vibrations)
- Also: T **must** be in Kelvins here
- The three comes from the fact that the molecules can move in three dimensions.

Average speed

- Since we know the average K is proportional to T $K \sim T$, then clearly the average speed should satisfy $\overline{v^2} \sim T$.
- However, there is a slight subtlety: Each molecule has equal chance to be moving in the positive or negative direction at each moment, so the **mean velocity is actually zero.**
- This signals that we should be using a different type of average instead of the mean.
- We'll use the **root mean square (rms)**

Root Mean Square

- Root Mean Square:

$$\boxed{v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}}}$$

- The bars represent the mean (average)
- Order of operations is important:
  - Square, then average, then square root.
  - If you do the operations in the wrong order, you will get zero.
- In particular, note that $\sqrt{\overline{v^2}} \neq \overline{\sqrt{v^2}}$.

RMS Values: Example

- Given – Particle 1: $v_x = 2\frac{m}{s}, \quad v_y = v_z = 0$

  Particle 2: $v_x = -2\frac{m}{s}, \quad v_y = v_z = 0$
- Then $\overline{v_x} = \overline{v_y} = \overline{v_z} = 0 \quad \rightarrow \quad \bar{v} = 0$
- But: $\overline{v_x^2} = \sqrt{\frac{1}{2}\left((2)^2 + (-2)^2\right)} = 4\frac{m}{s}$
- So: $v_{rms} = \sqrt{\overline{v^2}} = \boxed{2\ m/s}$

RMS Speed vs. Temperature

$$\bar{K} = \frac{1}{2}\mathrm{m}\overline{v^2} = \frac{1}{2}\mathrm{m}v_{rms}^2$$

$$\bar{K} = \frac{3}{2}kT$$

$$\boxed{v_{rms} = \sqrt{\frac{3kT}{m}}}$$

- Here, m is the mass per molecule.
- If you want to use mass per mole instead, the result is:

$$\boxed{v_{rms} = \sqrt{\frac{3RT}{M}}}$$

Once again, be careful: T must be in Kelvins

Example 1

- What is the (translational) kinetic energy of 3 moles of ideal gas at 300K?

- For one mole,

$$\overline{K} = \frac{3}{2}RT = \frac{3}{2}\left(8.314\frac{J}{mol \cdot K}\right)(300\ K) = \ 3741\frac{J}{mol}$$

- For three moles:

$$\overline{K} = (3\ mol)\left(\ 3741\frac{J}{mol}\right) = \boxed{\ 11{,}223\ J}$$

That's a lot of energy. Can we use it for anything? The 2nd law of thermodynamics says no! (See chapter 20)

Example 2

- What is rms speed for hydrogen molecules at STP?

$$m_{H2} = 2m_H = 2(1u)\left(\frac{1.674 \times 10^{-27}kg}{1\,u}\right) = 3.348 \times 10^{-27}\text{kg}$$

- So,

$$v_{rms} = \sqrt{\frac{3kT}{m_{H2}}} = \sqrt{\frac{3(1.38 \times 10^{-23})(300)}{3.348 \times 10^{-27}}} = \boxed{1927\,\frac{m}{s}}$$

- This is about 500 mph!
- You could also do with mass per mole.

Example 3

- Given: 4 moles of ideal gas in 50 L tank at 2 atm.

(a) What's the total translational kinetic energy of the gas?

Given: $n = 4$ moles, $V = (50\ L)\left(\frac{10^{-3} m^3}{1\ L}\right) = .05\ m^3$,

$$P = (2\ atm)\left(\frac{1.013 \times 10^5 Pa}{1\ atm}\right) = 2.026 \times 10^5 Pa$$

$$PV = nRT \quad \rightarrow \quad T = \frac{PV}{nR} = \frac{(2.026 \times 10^5)(.05)}{4(8.314)} = 304.6\ K$$

$$K = n(K\ per\ mole) = n\left(\frac{3}{2} RT\right)$$

$$= (4)\left(\frac{3}{2}\right)(8.314)(304.6) = \boxed{15{,}194.7\ J}$$

Example 3 (continued)

(b) v_{rms} is measured and found to be 1378.16 m/s. What type of molecule is it? (Assume it is monatomic)

$$v_{rms} = \sqrt{\frac{3RT}{M}} \quad \rightarrow \quad \text{Solve for M}$$

$$M = \frac{3RT}{v_{rms}^2} = \frac{3(8.314)(304.6)}{(1378.16)^2} = .004kg = 4\ g$$

This is the mass of helium nucleus → Gas is helium

Example 4

Consider nitrogen gas N_2 at STP. If the average distance between molecules is found to be $\bar{d} = 4\,nm$, what is the average time between molecular collisions?

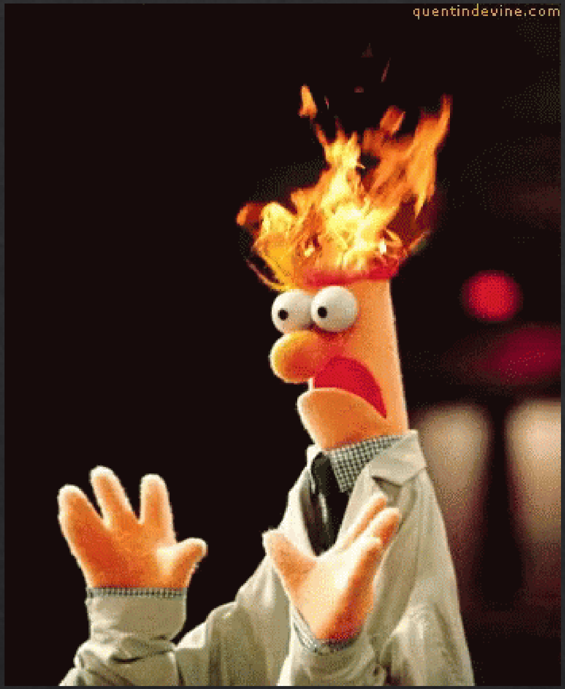
$$m = 2u = 4.68 \times 10^{-26} kg \quad \text{(per molecule)}$$

$$v_{rms} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3(1.38 \times 10^{-23})(273)}{4.68 \times 10^{-26}}} = 509\,m/s$$

$$\Delta t = \frac{\bar{d}}{v_{rms}} = \frac{4 \times 10^{-9} m}{509\,s} = \boxed{7.86 \times 10^{-12} s}$$

Number of collisions per time (per molecule):

$$\frac{1}{\Delta t} = 1.27 \times 10^{11} \text{ per second}$$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Ch. 19 – Heat and the 1st law of thermodynamics

Heat

- **Heat:** $Q =$ transfer of energy due to temperature differences.
- The energy is transferred by elastic collisions between moving molecules.
- Temperature is a measure of average kinetic energy per molecule, so as heat flows the higher temp. material cools, the lower temp. material heats up.
- **This chapter:** we want to study heat in more detail and its connection to work.

Units of heat

- Since heat is an energy change, we can measure it in units of energy.
- Our primary unit for heat will be **Joules**.
- Often **calories** will also be used:

$1\ calorie = 4.18\ J$ = energy needed to raise 1 g of water by 1°C

- **Be careful:** the "calories" listed on food packages are actually kilocalories (kcal)

Example 0

- It requires 3500 kcal of work to burn 1 pound of fat. Running consumes approximately 1000 kcal (4.18×10^6 J) of energy per hour for a 65 kg person.
- How far does a person have to run at a speed of 7.5 mph to burn off a pound of fat?
- Given:
  - $W = 3500\ kcal$
  - Power: $P = 1000\ \frac{kcal}{hr}$ = rate at which work is done (metabolic rate)
  - Speed: $v = 7.5 \frac{miles}{hour}$
- Want: $x =$ distance

Example 0 (continued)

- First find the time needed to burn the calories:

$$P = \frac{W}{\Delta t} \quad \rightarrow \quad \Delta t = \frac{W}{P} = \frac{3500\ kcal}{1000\ kcal/hr} = 3.5\ hrs$$

- Now find distance:

$$\Delta x = vt = \left(7.5\frac{miles}{hr}\right)(3.5\ hrs) = \boxed{26.25\ miles}$$

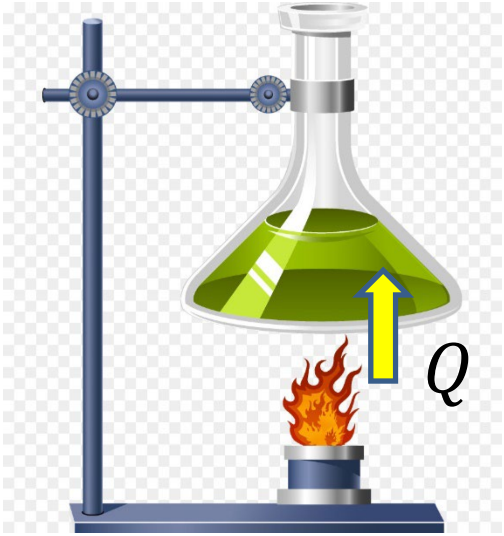
- Basically, you have to run a full marathon to burn one pound of fat!
- Reducing food intake is a much more efficient method of losing weight.

Heat & Temperature

- Be careful not to confuse heat with temperature
- In particular:
  - Temperature is an **intensive** variable (it does not depend on the amount of material)
  - Heat is **extensive** (it is proportional to the amount of material)
- Also, temp. is an actual ***energy*** (per molecule) but heat is an ***energy*** ***change***.

Effects of Heat

- When heat is added to a material it can cause several things to happen:
  - Temperature change
  - Phase change
  - Expansion and work



- We’ll look at each of these in turn, starting with temperature changes.

Specific Heat

- How much heat Q needs to be added to an object to produce a given temperature change, ΔT?
- Answer: $\boxed{Q = mC\Delta T}$
- $m =$ mass of the object
- $C =$ specific heat (in $\frac{J}{kg \cdot K}$ or $\frac{cal}{kg \cdot K}$)
- Note that only differences in T enter here, so either K or °C can be used.
- The product mC is the **heat capacity**

A table of specific heats for different materials can be found in the book

Example 1

- $Q = 2000\ J$ of heat are added to a pot of water with mass $m = 2\ kg$ and initial temp. $T_0 =$ 5°C. What is final T?

- For water, $C = 4180\ \frac{J}{kg \cdot K}\ \left(= 4180\ \frac{J}{kg \cdot {}^\circ\mathrm{C}}\right)$

$$Q = mC\Delta T \quad \rightarrow \Delta T = \frac{Q}{mC} = \frac{2000\ J}{(2\ kg)(4180\ \frac{J}{kg\ \cdot\ {}^\circ\mathrm{C}})} = .24\ {}^\circ\mathrm{C}$$

- Final temp.: $T = T_0 + \Delta T = 5 + .24 = \boxed{5.24{}^\circ\mathrm{C}}$

$(= 278.24\ K)$

Example 2

- One gram of gin $\left(C_G = 2400\,\frac{J}{kg\cdot°C}\right)$ at temp. $T_G = 20°\text{C}$ is mixed with 2 g of tonic water at $T_W = 10°\text{C}$. What is the final temperature of the mixture?
- Use fact that the energy gained by the water equals the energy lost from the gin:

$$m_W C_W (T - T_W) = m_G C_G (T_G - T) \rightarrow \text{ Solve for final T}$$

$$(m_W C_W + m_G C_G)T = m_G C_G T_G + m_W C_W T_W$$

$$T = \frac{m_G C_G T_G + m_W C_W T_W}{(m_W C_W + m_G C_G)}$$

$$= \frac{(.001)(2400)(20) + (.002)(4180)(10)}{\big((.002)(4180) + (.001)(2400)\big)} = \boxed{12.23°\text{C}}$$

Example 3

- A copper cup of mass .1 kg at 20°C is filled with .2 kg of coffee at 70°C . Final temperature?
- Copper: $m_{cu} = 0.1\ kg, C_{cu} = 390\ \frac{J}{kg°C}, T_{cu} = 20°C$
- Water: $m_w = 0.2\ kg, C_w = 4180\ \frac{J}{kg°C}, T_w = 70°C$

$$m_{cu}C_{cu}(T - T_{cu}) = m_W C_W (T_W - T)$$

$$T = \frac{m_{cu}C_{cu}T_{cu} + m_W C_W T_W}{(m_{cu}C_{cu} + m_w C_w)}$$

$$= \frac{(.1)(390)(20) + (.2)(4180)(70)}{(.1)(390) + (.2)(4180)} = \boxed{67.8°C}$$

Latent Heat

- Heat can also cause a phase change:
  - Melting, vaporizing, combusting, etc.
- The heat required to cause this phase change is:

$$\boxed{Q = \pm mL}$$

- L is the **latent heat**, or heat of transformation
- Units of L: J/kg or cal$/kg$
- Notice that you have to explicitly put in the correct sign:
  - + if energy is added (melting, evaporating),
  - - if energy removed (freezing, condensing)

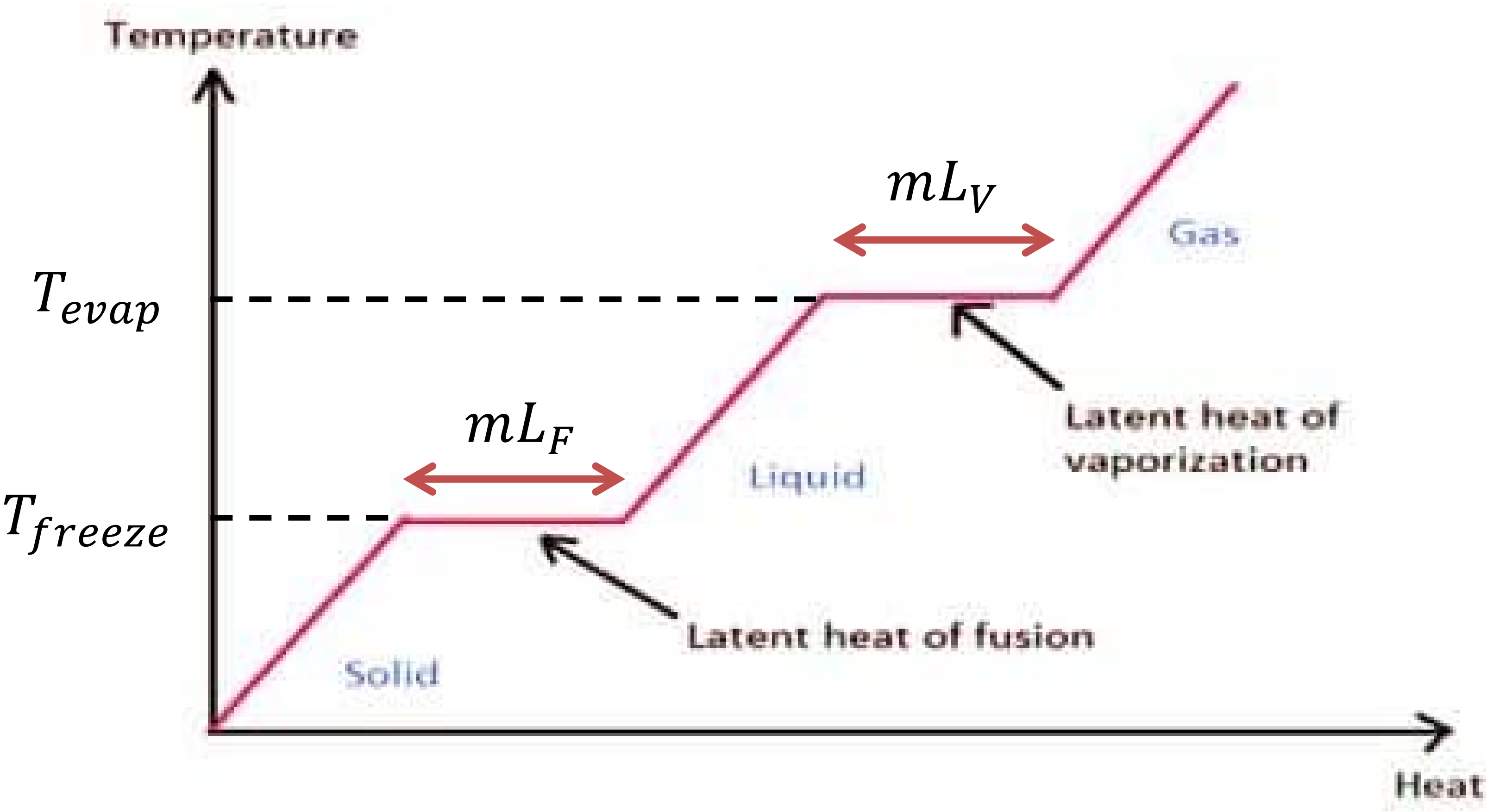
Latent Heat

Latent heats come in three kinds:

- L_V = latent heat of vaporization (evaporation/condensation)
- L_F = latent heat of fusion (melting/freezing)
- L_C = latent heat of combustion (burning)

- Values of L for a given material can be found in a table in the book.

Latent and Specific Heats



- The slopes of the rising parts are given by $^1/_{mC}$, with the appropriate specific heat C . $(\frac{\Delta T}{Q} = 1/\mathrm{mC})$

Example 4

How much heat is required to turn a 5 kg block of ice at 0°C into water vapor at 100°C ?

$$Q = (heat\ to\ melt) + (heat\ to\ raise\ T) + (heat\ to\ evaporate)$$

$$= mL_F + mC\Delta T + mL_V = \mathrm{m}(L_F + C\Delta T + L_V)$$

$$= (5\ \mathrm{kg})\left[3.33 \times 10^5\ {}^{J}/_{kg} + (4180\ {}^{J}/_{kg\,°\mathrm{C}})(100°\mathrm{C}) + 22.6 \times 10^5\ {}^{J}/_{kg}\right]$$

$$= \boxed{1.51 \times 10^7\ J}$$

Example 5

How much ice at $-20°\mathrm{C}$ must be added to .25kg of water at $+20°\mathrm{C}$ in order for final temperature to be $0°\mathrm{C}$, with all the ice melted?

$(Q_{ice} > 0,\ \ Q_{water} < 0)$

$$Q_{ice} + Q_{water} = 0$$

$$(\ m_{ice} C_{ice} \Delta T_{ice} + m_{ice} L_F) + m_w\ C_w \Delta T_w = 0$$

$$m_{ice} = \frac{-\ m_w C_w \Delta T_w}{C_{ice} \Delta T_{ice} + L_F}$$

$$= \frac{-(.25)(4180)\ (-20°\mathrm{C})}{(2100)(+20°\mathrm{C}) + 3.33 \times 10^5}$$

$$= \ .0557 kg \ = \boxed{55.7 g}$$

Example 6

A lantern burns gasoline ($L_C = 4.6 \times 10^7\ {}^{J}/_{kg}$) and emits as much light as a 25 W light bulb. How much gas is burned in 10 hours?

$1\,W = 1\,J/s$

$$Q = (power)\Delta t = (25\,J/s)(10\,hrs)(3600\,s/hr)$$

$$= 9 \times 10^5 J$$

But: $Q = m\,L_C$

So:

$$m = \frac{Q}{L_C} = \frac{9 \times 10^5 J}{4.6 \times 10^7\ {}^{J}/_{kg}} = .0196\,kg = \boxed{19.6\,g}$$

Work and Volume

- Recall that $W = \int F(x)\,dx$.
- But pressure and volume are given by

$$P = \frac{F}{A}, \qquad V = Ax$$

- Therefore: $W = \int \left(\frac{F}{A}\right)(A\,dx) = \int P\,dV$
- So we have an alternate formula for work:

$$\boxed{W = \int P(V)\,dV}$$

- If P is constant: $\boxed{\text{W} = P\Delta V}$

But **ONLY** if P is constant!

Internal Energy

- Recall that T measures the **translational** kinetic energy per molecule.
- But there can also be kinetic energy locked up in **rotations** of molecules or internal **vibrations** of the atoms inside them.
- The **internal energy** E includes **all** the kinetic energy (rotational + translational + vibrational)
- Internal energy gives the KE of the *entire material* (it is **not** per molecule or per mole)
- In other words, it is an extensive variable, unlike the temperature (which is an intensive variable)

First law of thermodynamics

- The first law of thermodynamics is an expression of conservation of energy. It relates the heat, work, and internal energy:

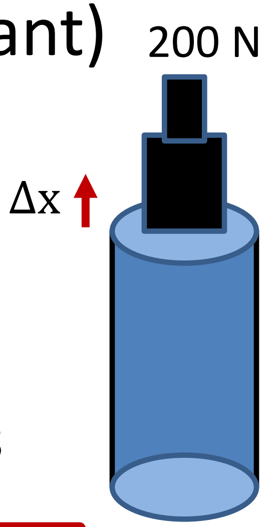
$$\boxed{Q = W + \Delta E}$$

- What this means is that when you add heat:
  - Part of it goes into increasing the energy of the molecules (temp. and phase changes). This is described by ΔE.
  - The remainder goes into doing work, expanding against the surrounding environment.

Example 7

- A vertical rod supports 200 N of weight.

(a) If the rod expands by 1mm in length, how much work is done? (Cross section A=0.1 m^2 stays constant)



Assuming pressure is constant:

$$P = \frac{F}{A} = \frac{200\ N}{.1\ m^2} = 2000\ Pa$$

$$\Delta V = A\Delta x = (.1\ m^2)(.001\ \mathrm{m}) = 10^{-4} m^3$$

$$W = P\Delta V = (2000\ \mathrm{Pa})(10^{-4} m^3) = \boxed{0.2\ \mathrm{J}}$$

-or-

$$W = F\Delta \mathrm{x} = (200\ \mathrm{N})(10^{-3} m) = 0.2\ \mathrm{J}$$

Example 7 (continued)

(b) If .5 J of heat were added during the expansion, what is the change in internal energy of the material?

$$\Delta E = Q - W = .5\,J - .2J = \boxed{.3\,J}$$

Example 8

(a) How much work is needed to blow up a balloon of radius r = .2m?

Pressure: $P = 1\,atm = 1.015 \times 10^5\,Pa = constant$

Initial and final volumes:

$$V_0 \approx 0, \qquad V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(.2)^3 = .034\,m^3$$

So: $\Delta V = .034\,m^3$

$$W = P\Delta V = (1.015 \times 10^5\,Pa)(.034\,m^3) = \boxed{3444.2\,J}$$

$$(= 823.97\,cal)$$

Example 8 (continued)

- The balloon cools as it expands, losing 1000 J of heat to its surroundings.
- How much does the internal energy of the balloon +gas change?

$$\Delta E = Q - W = (-1000\,J) - 3444.2J$$

$$= \boxed{-4444.2\,J}$$

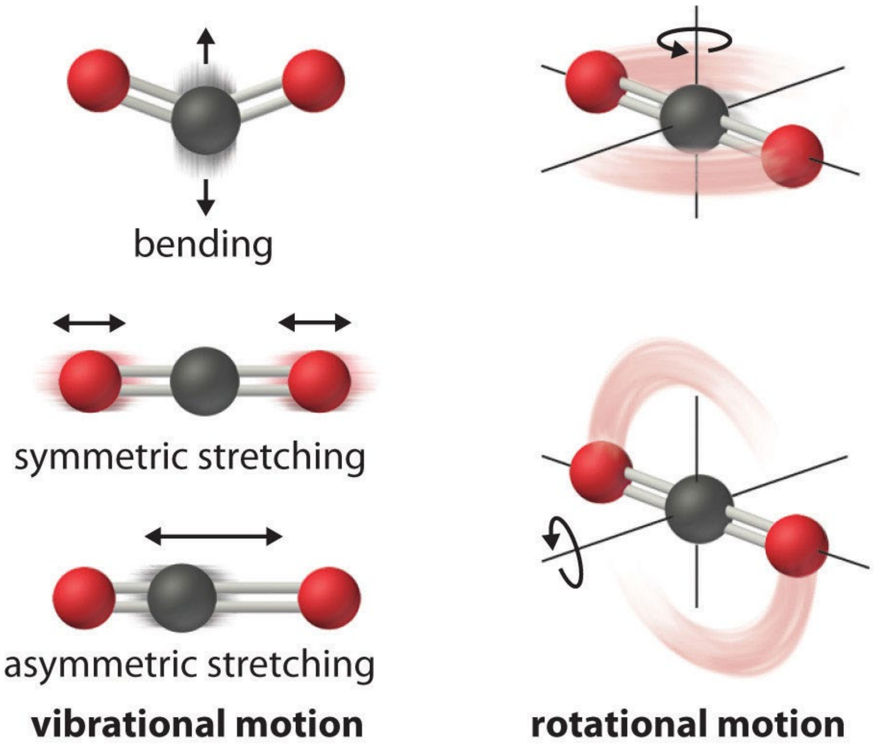
Internal energy of ideal gas

- For an ideal gas, the internal energy and the temperature are directly proportional to each other.
- For a monatomic gas (each molecule is a single atom):

$$\boxed{E = \frac{3}{2}NkT = \frac{3}{2}nRT}$$

- $\frac{3}{2}kT$ and $\frac{3}{2}RT$ are respectively the energy per molecule and per mole.
- What about molecules composed of more than one atom?

Degrees of freedom


bending
symmetric stretching
asymmetric stretching
vibrational motion
rotational motion

- For the monatomic molecule, the atom can move in three directions through space (along x, y, and z axes)
- That was the source of the 3 in the previous equations.
- For a molecule made up of more than one atom, additional movements are possible, since the molecule can rotate or vibrate:

Degrees of freedom

- The number of degrees of freedom d is the number of different types of motion that the molecule can exhibit.
- d = # of (translations)+(rotations)+(vibrations)
- At room temperature, there may not be enough energy to excite some degrees of freedom, so they aren't counted.
- In general, at room temperature:
  - Monatomic: d=3 (translational)
  - Diatomic: d=5 (3 translational + 2 rotational)
  - >2 atoms: depends on specific shape of molecule

Equipartition theorem

- The equipartition theorem says that internal energy is distributed equally among the available degrees of freedom
- Each degree of freedom in a molecule will have energy $\frac{1}{2}kT$ per molecule (or $\frac{1}{2}RT$ per mole).
- For an ideal gas (no longer necessarily monatomic), this means:

$$\boxed{E = \frac{d}{2}NkT = \frac{d}{2}nRT}$$

- **Warning:** This is for ideal gas **only**.

Notice E depends **only** on temperature for an ideal gas. This is **not** true for other materials, where E may depend on P or various other quantities.

Example 9

- What is the internal energy of 1 mole of an ideal monatomic gas at STP?
- $n = 1\,mol,\ R = 8.315\ \frac{J}{mol\cdot K},\ T = 273\,K,\ d = 3$
- Internal energy:

$$E = \frac{d}{2}nRT = \frac{3}{2}(1)(8.315)(273) = \boxed{34{,}129\,J}$$

Warning: Be careful to distinguish between internal energy, $E = \frac{d}{2}nRT$, and **changes** in internal energy, $\Delta E = \frac{d}{2}nR\Delta T$.

Example 10

- 0.2 moles of a monatomic ideal gas expand from $V_0 = 2\,m^3$ to $V = 4\,m^3$, with pressure held constant.
- If the initial temperature is $T_0 = 300\,K$, find:

(a) The work done.

Pressure: $P = \frac{nRT}{V_0} = \frac{(.2)(8.315)(300)}{2} = 249.42\,Pa$

$$W = P\Delta V = (249.42)(4-2) = \boxed{498.84\,J}$$

(b) The final temperature.

$$\frac{V_0}{T_0} = \frac{V}{T} \quad \rightarrow \quad T = \left(\frac{V}{V_0}\right) T_0 = \left(\frac{4}{2}\right)(300\text{ K}) = \boxed{600\text{ K}}$$

Example 10 (continued)

(c) The internal energy change.

$$\Delta E = \frac{3}{2} nR\Delta T = \frac{3}{2}(.2)(8.315)(300) = \boxed{748.3\ J}$$

(d) The heat added or removed.

$$Q = W + \Delta E = 748.3 + 498.8 = \boxed{1247.1\ J}$$

(e) Which of the previous answers changes for a diatomic molecule (d=5)

- (a) & (b) are independent of d. They don't change.
- (c) becomes: $\Delta E = \frac{5}{2} nR\Delta T = \ 1247.1\ J$
- Since ΔE increased, the heat needed in (d) will increase:

$$Q = 1745.9\ J$$

Example 11

(a) Given n=1.75 moles of ideal gas, consider an **isothermal** expansion from $V_0 = .03m^3$ to V $= .06m^3$ at $T = 310\ K$. Find the work done and heat added.

First, notice that $\Delta E = 0$, since $\Delta T = 0$. Then:

$$W = \int_{V_0}^{V} P(V)\,dV = \int_{V_0}^{V} \frac{nRT}{V}\,dV = nRT \int_{V_0}^{V} \frac{dV}{V}$$

$$= nRT \ln \frac{V}{V_0} = (1.75)(8.315)(310) \ln \frac{.06}{.03} = \boxed{3125\ J}$$

$$Q = \Delta E + W = 0 + 3125 = \boxed{3125\,J}$$

Example 11 (continued)

(b) Consider the same gas, now undergoing an **isobaric** (constant pressure) expansion between the same volumes. Find the work and internal energy change.

$$W = \int P\,dV = P\Delta V = (1.5 \times 10^5)(.06 - .03)$$

$$= \boxed{4.5 \times 10^3\ J}$$

$$\Delta E = Q - W = (11.26 - 4.5) \times 10^3\ J = \boxed{6.76 \times 10^3\ J}$$

ONE-DIMENSIONAL MOTION

Ch. 2 Motion along a line

Kinematics

- Mechanics, the study of motion, splits into two parts:
  - **Kinematics:** the **description** of motion
    - *Given acceleration*, find equations for position and velocity as functions of time
  - **Dynamics:** the **explanation** of motion
    - What *causes* the acceleration? (Answer: forces)
- This chapter and the next: kinematics
- Chapter 4: the beginning of dynamics

Notation

- t = time (in seconds)
- x = position (usually in meters) at time t
- t_0 = initial time (will usually take $t_0 = 0$)
- x_0 = initial position at time t_0

- Δ (capital Greek delta) = “Change in”

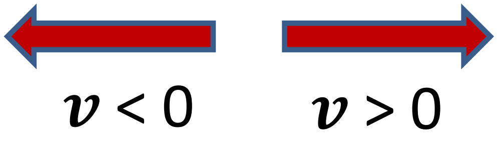
  = (final value) – (initial value)

- For example:
  - $\Delta x = x - x_0$, and $\Delta t = t - t_0$

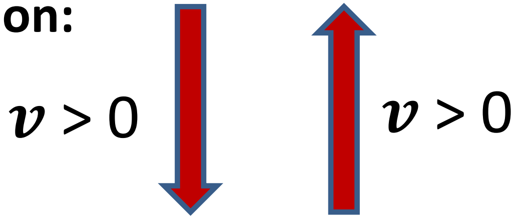
Speed and velocity

- **Velocity** $\boldsymbol{v}$ is the **rate of change** of position, i.e. the change in position per unit time
- Units of $\boldsymbol{v}$ are meters per second (m/s)
- Velocity can be positive or negative. The sign indicates direction:
  - – Positive $\boldsymbol{v}$ means motion upward or to the right
  - – Negative $\boldsymbol{v}$ means motion downward or to the left

Horizontal motion:


$\boldsymbol{v} < 0$
$\boldsymbol{v} > 0$

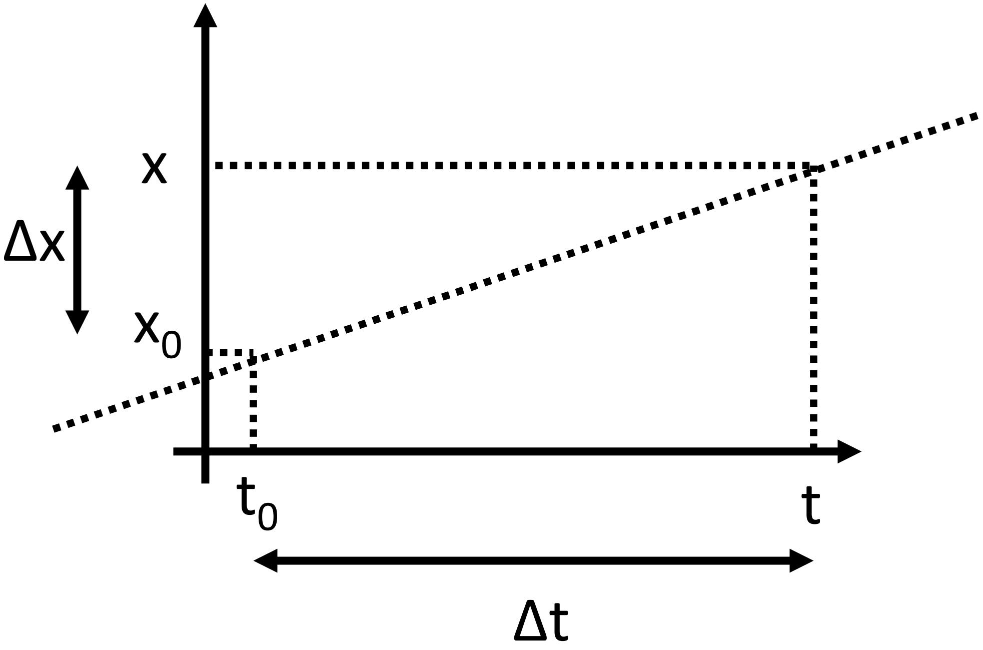
Vertical motion:


$\boldsymbol{v} > 0$
$\boldsymbol{v} > 0$

Speed and velocity

- The sign of velocity gives the **direction** object is moving. The absolute value gives **how fast** it is moving.
- This absolute value is the **speed**: $\overset{\text{speed}}{\overset{\downarrow}{v}} = |\overset{\text{velocity}}{\overset{\downarrow}{\mathbf{v}}}|$
- Note that the velocity is bold print, the speed is not. (More on the meaning of this in the next chapter.)
- To reiterate:
  - – Velocity can be negative (sign indicates direction)
  - – Speed is always positive (or zero)

Average Velocity



- Consider a time interval, from t_0 to t.
- What is the average velocity over this interval?

Average velocity: $\bar{\boldsymbol{v}} = \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t - t_0}$ (bar = average)

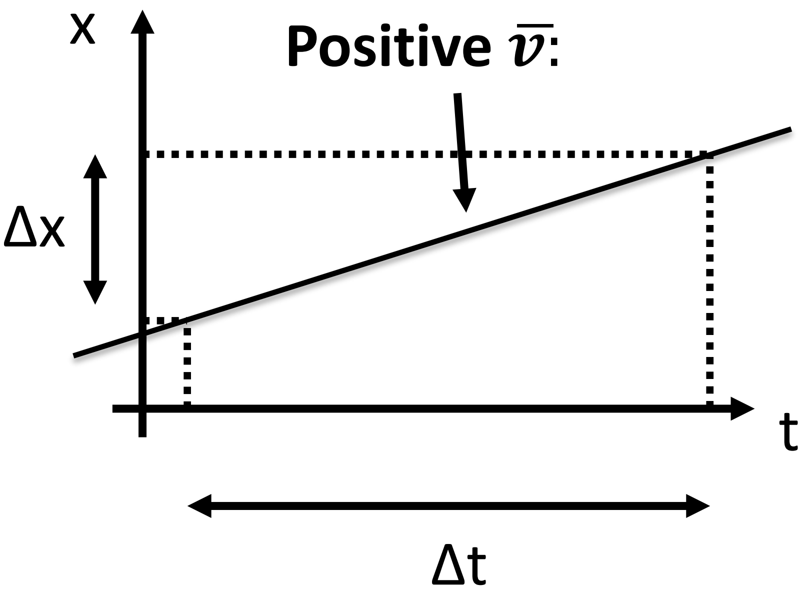
Average speed: $|\bar{\boldsymbol{v}}| = \left|\frac{\Delta x}{\Delta t}\right| = \left|\frac{x - x_0}{t - t_0}\right|$

Average Velocity

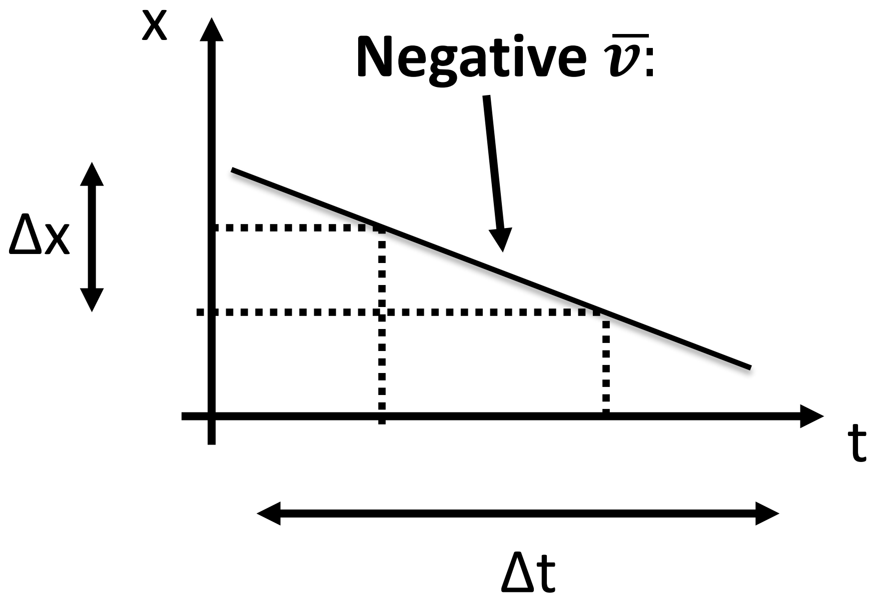
Average velocity:

$$\bar{\boldsymbol{v}} = \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t - t_0}$$

- This is the **slope** of the straight line joining the initial and final times


x
Positive $\bar{v}$:
Δx
t
Δt

Object is moving to right (on average)

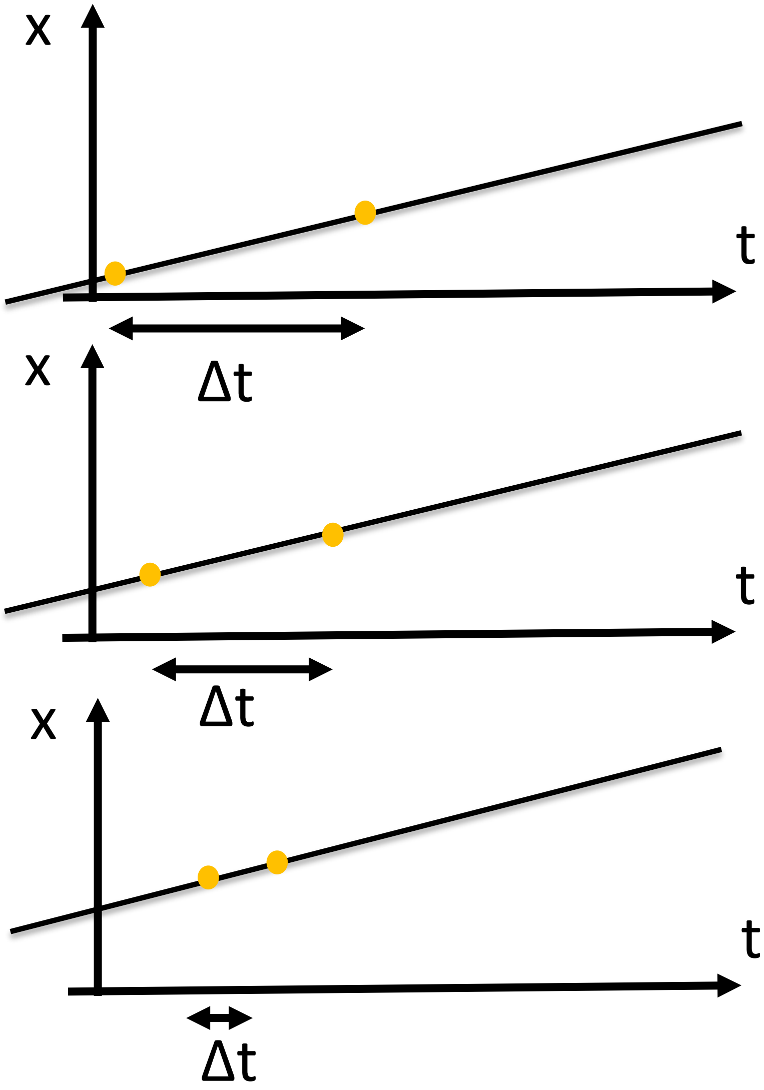

x
Negative $\bar{v}$:
Δx
t
Δt

Object is moving to left (on average)

Instantaneous Velocity

- **Instantaneous velocity**, $\boldsymbol{v}$, is the velocity at a particular moment, not average over an interval of time. It is a function of time: $\boldsymbol{v(t)}$
- Slide the endpoints of the interval closer together so that Δx and Δt become infinitesimal:

$$\Delta x \rightarrow dx\,, \qquad \Delta t \rightarrow dt$$

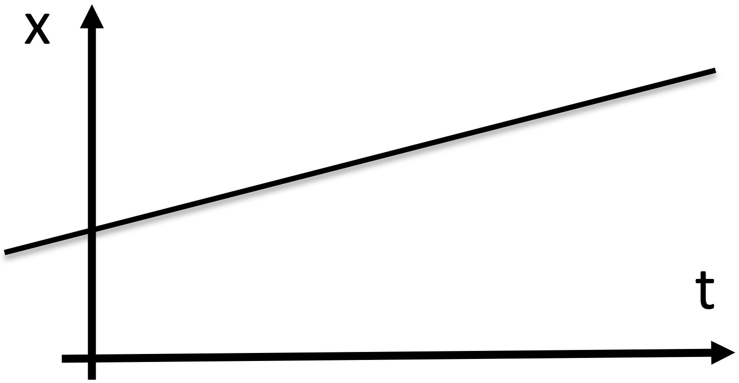


Instantaneous Velocity

- As you take the infinitesimal limit, the ratio Δx/Δt becomes a **derivative.**
- **Instantaneous velocity:**

$$\boxed{\boldsymbol{v} = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}}$$

- The instantaneous velocity measures the slope of the line **tangent** to the curve at time t



Average & Instantaneous Velocities

- Note that:
  - $\bar{v}$ is the average over some finite **interval**, Δt.
  - v is value at a **precise moment**, t.
- When people say “velocity” without any qualifier, they usually mean the instantaneous velocity.
- If position x(t) is known as function of time, v(t) is found by taking derivative.
- We want to minimize use of calculus for now, so we will focus mostly on cases where we can find velocity without taking derivatives.

Example 1

- A jogger travels 400 m in 2 minutes. What is his average speed in meters per second?

$$|\overline{\boldsymbol{v}}| = \left|\frac{\Delta x}{\Delta t}\right| = \left|\frac{400\ m}{2\ min}\right| \cdot \underset{\substack{\big\uparrow \\ \text{Conversion factor}}}{\left(\frac{1\ min}{60\ s}\right)} = \boxed{3.33\ \frac{m}{s}}$$

- Note that we are given no information about direction, so we can only find **speed**, not **velocity**.

Example 2

- An airplane travels at 500 mph (average speed). How far (in miles) does it travel in 20 minutes?
- Given quantities: $|\overline{\boldsymbol{v}}| = 500\ \frac{miles}{hour}$

$$\Delta t = 20\ min.\left(\underset{\substack{\uparrow \\ \text{Conversion factor}}}{\frac{1\ hr}{60\ min.}}\right) = .333\ hr$$

- Rearrange $|\overline{\boldsymbol{v}}| = \left|\frac{\Delta x}{\Delta t}\right|$ to solve for distance:

$$|\Delta x| = |\overline{v}\,\Delta t| = \left(500\ \frac{miles}{hr}\right)(.333\ hr) = \boxed{167\ miles}$$

Acceleration

- One more basic kinematic quantity is needed: **Acceleration** is the rate of change of the velocity:
  - Average acceleration: $\boxed{\bar{\boldsymbol{a}} = \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{\boldsymbol{v} - \boldsymbol{v}_0}{t - t_0}}$
  - Instantaneous acceleration: $\boxed{\boldsymbol{a}(\boldsymbol{t}) = \frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}}$
- Just as velocity was the slope of x vs. t graph, acceleration is the slope of the v vs. t graph.
- Units of acceleration: $\frac{m}{s^2}$ (or $\frac{ft}{s^2}$, $\frac{mile}{hr^2}$, etc.)

Acceleration

- The sign of acceleration tells the direction of change of velocity:
  - $\boldsymbol{a} > 0 \quad \rightarrow$ velocity is increasing
  - $\boldsymbol{a} < 0 \quad \rightarrow$ velocity is decreasing
- Note that the previous lines say **velocity**, not **speed**.
- If velocity is negative, then:
  - $\boldsymbol{a} > 0 \quad \rightarrow \quad \boldsymbol{v}$ is increasing (becoming less negative), so speed is decreasing (slowing down)
  - $\boldsymbol{a} < 0 \quad \rightarrow \quad \boldsymbol{v}$ is decreasing (becoming more negative), so speed is decreasing (speeding up)

Position, velocity, and acceleration

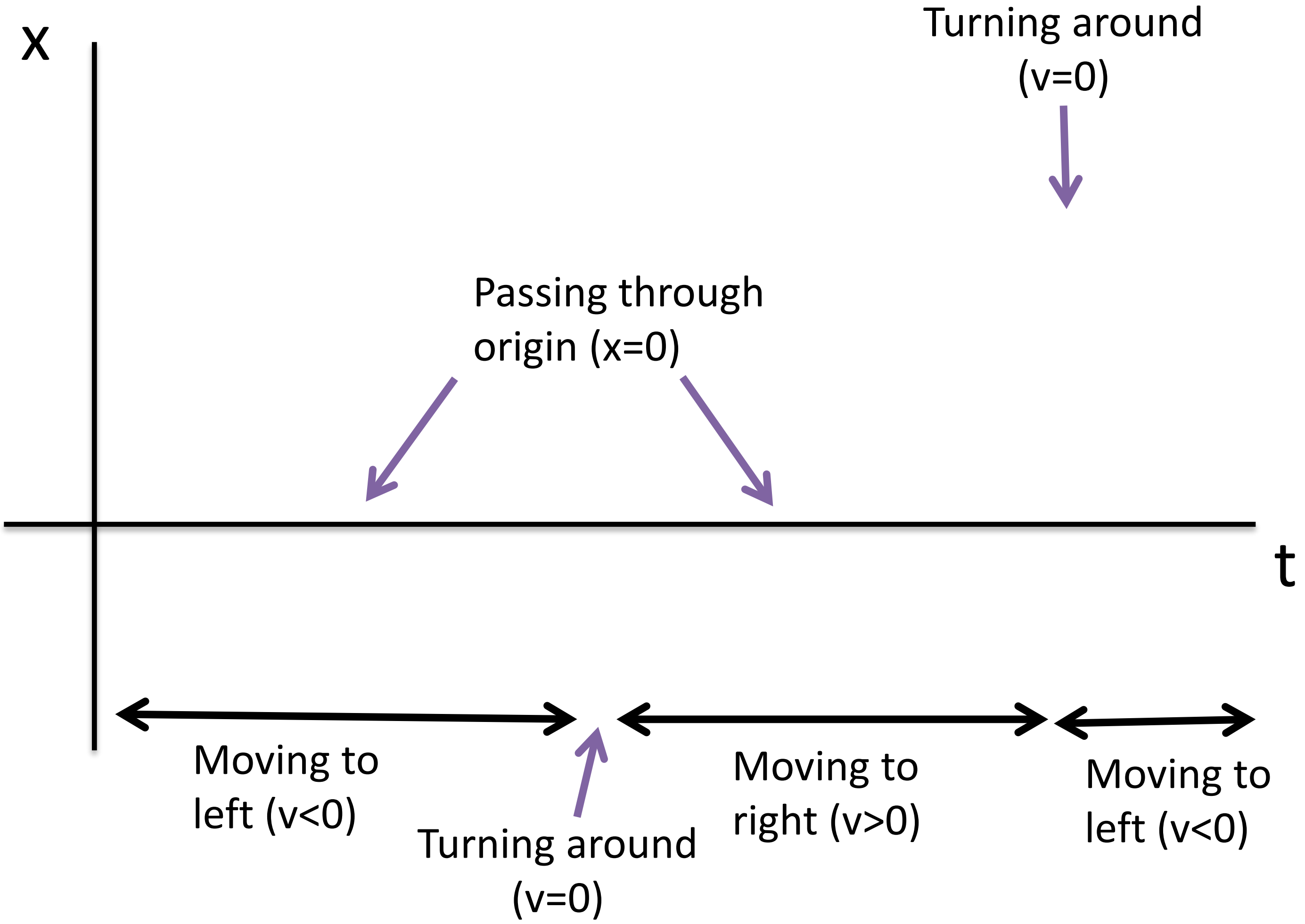
- We now have all of our basic kinematic quantities: position, velocity, and acceleration.
- In general, all three are time-dependent.
- Summarizing their relations:

$\boldsymbol{v}(\boldsymbol{t}) = \frac{dx}{dt}$ = rate of change of x = slope of x vs. t graph

$\boldsymbol{a}(\boldsymbol{t}) = \frac{d\boldsymbol{v}}{dt} = \frac{d^2x}{dt^2} =$ rate of change of v = slope of v vs. t graph

= rate at which slope of x vs. t graph is changing (curvature)

x versus t graph



Turning around (v=0)

Passing through origin (x=0)

Moving to left (v<0)

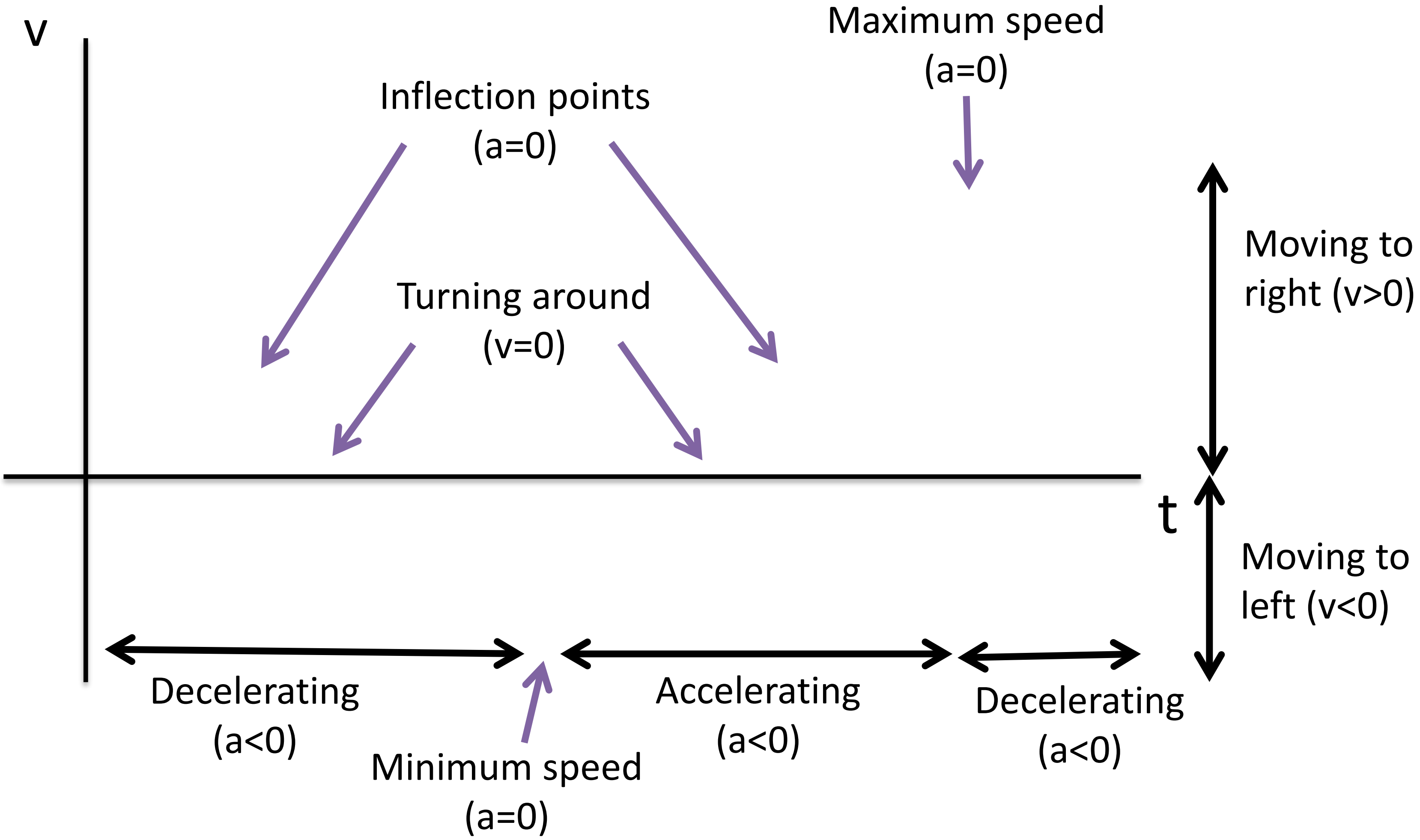
Turning around (v=0)

Moving to right (v>0)

Moving to left (v<0)

- **Important point:** $v = 0$ at turning points (maxima and minima)

v versus t graph



Maximum speed
(a=0)

Inflection points
(a=0)

Turning around
(v=0)

Moving to
right (v>0)

Moving to
left (v<0)

Decelerating
(a<0)

Minimum speed
(a=0)

Accelerating
(a<0)

Decelerating
(a<0)

- a=0 at inflection points, where graph switches between concave up and concave down

Example 1

- A car accelerates from 0 to 60 mph in 7 seconds. What is its average acceleration in ft/s^2 ?
- Known: $\boldsymbol{v}_0 = 0, \qquad \Delta t = 7\ s$

$$\boldsymbol{v} = \left(60\,\frac{mile}{hr}\right)\left(\frac{5280\ ft}{1\ mile}\right)\left(\frac{1\ hr}{3600\ s}\right) = 88\ \frac{ft}{s}$$

- So: $\bar{\boldsymbol{a}} = \dfrac{\Delta \boldsymbol{v}}{\Delta t} = \dfrac{\boldsymbol{v} - \boldsymbol{v}_0}{\Delta t} = \dfrac{(88-0)\,{}^{ft}/_{s}}{7\ s} = \boxed{12.57\ \dfrac{ft}{s^2}}$

The fine print:

- We assumed that the motion is to right, so we could take $v > 0$ and not worry about minus signs.
- If we are told that a is constant, then the bar can be dropped: if something is constant, then its average and instantaneous values are the same.

Example 2

- Same car as last example now decelerates from $88\ \frac{ft}{s}$ to rest in 4 seconds. What is its average acceleration in ft/s^2 ?
- Known: $\boldsymbol{v}_0 = 88\ \frac{ft}{s}, \qquad \Delta t = 4\ s, \quad \boldsymbol{v} = 0$
- So:

$$\bar{\boldsymbol{a}} = \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{\boldsymbol{v} - \boldsymbol{v}_0}{\Delta t} = \frac{(0 - 88)\ {}^{ft}\!/_{s}}{4\ s} = \boxed{-22.0\ \frac{ft}{s^2}}$$

- The car is now slowing down (velocity and speed are decreasing)

Special Case: Constant Acceleration

- Now we assume **a = constant** (not a function of time). we can write explicit formulas for position and velocity as functions of time.
- **Constant acceleration formulas:**

1. $x = x_0 + v_0 t + \frac{1}{2} a t^2$
2. $v = v_0 + a t$
3. $v^2 = v_0^2 + 2a(x - x_0)$

x and v are position and velocity at time t
x_0 and v_0 are position and velocity at time 0

Constant acceleration equations

1. $x = x_0 + v_0 t + \frac{1}{2} a t^2$
2. $v = v_0 + at$
3. $v^2 = v_0^2 + 2a(x - x_0)$

- Note:
  - Differentiating 1st equation gives 2nd one. Differentiating again gives a.
  - Integrating a once gives eq. 2 and integrating again gives eq. 1. x_0 and v_0 are integration constants
  - 3rd equation follows from plugging eq. 2 into eq. 1 to eliminate t.
  - These hold **only** if a = constant.
  - Note that *distance = (speed) x (time)* is true **only** if $a = 0$.

Example 1

- A skier descends slope. with initial speed $v_0 = 8\ \frac{m}{s}$. She decelerates *at a constant rate*, coming to a stop after a distance of $\Delta x = x - x_0 = 400\ m$.

(a) What is her acceleration?

$v^2 = v_0^2 + 2a(x - x_0)$ → Solve for a.

$$a = \frac{v^2 - v_0^2}{2(x - x_0)} = \frac{0^2 - (8\frac{m}{s})^2}{2(400\ m)} = \boxed{-.08\ \frac{m}{s^2}}$$

(b) How long does it take her to stop?

$$v = v_0 + at \quad \rightarrow \quad t = \frac{v - v_0}{a} = \frac{0 - (8\ m/s)}{.8\ m/s^2} = \boxed{100\ s}$$

Example 2

- Initially, car has $v_0 = -15\ \frac{m}{s}$, but it accelerates at $a = +2\ \frac{m}{s^2}$.

(a) At what time does car turn around?

$$v = v_0 + at \quad \rightarrow \quad t = \frac{v - v_0}{a} = \frac{0 - (-15\frac{m}{s})}{2\frac{m}{s^2}} = \boxed{7.5\ s}$$

v = 0 at turning point

(b) Where is it at this time? (Assume $x_0 = 0.$)

$$x = x_0 + v_0 t + \frac{1}{2}at^2 = 0 + (-15)(7.5) + \frac{1}{2}(2)(7.5)^2 = \boxed{-56.25\ m}$$

(c) At what time will it return to the origin?

$$\cancel{x} = \cancel{x_0} + v_0 t + \frac{1}{2}at^2 \quad \rightarrow \quad 0 = \left(v_0 + \frac{1}{2}at\right)t$$

Solutions: (i) t=0 (We already knew this)

(ii) $t = -\frac{2v_0}{a} = -\frac{2(-15)}{2} = \boxed{15s}$

Example 3

- A Porsche brakes from $v_0 = 75\,\frac{km}{hr}$ $\left(= 20.83\ \frac{m}{s}\right)$ to $v = 45\,\frac{km}{hr}$ $\left(= 12.5\ \frac{m}{s}\right)$, *with constant acceleration*. During this time, the car travels distance $x - x_0 = 88\ m$.

(a) What is the acceleration?

$$v^2 = v_0^2 + 2a(x - x_0) \qquad \rightarrow \qquad \text{Solve for a.}$$

$$a = \frac{v^2 - v_0^2}{2(x - x_0)} = \frac{(12.5\,\frac{m}{s})^2 - (20.83\,\frac{m}{s})^2}{2(88\ m)} = \boxed{-1.58\ \frac{m}{s^2}}$$

(b) How long did it take to reach final speed?

$$v = v_0 + at \qquad \rightarrow \qquad t = \frac{v - v_0}{a} = \frac{(12.5\,\frac{m}{s}) - (20.83\,\frac{m}{s})}{-1.58\,\frac{m}{s^2}} = \boxed{5.27\ s}$$

Example continued ➡

Example 3 (continued)

(c) If the acceleration continues, how long does it take to bring car to rest?

We know: $v = 0, \quad v_0 = 75\ \frac{km}{hr} = 20.83\ \frac{m}{s}, \qquad a = -1.58\ \frac{m}{s^2}$

So:

$$\text{v} = v_0 + \text{at} \qquad \rightarrow \qquad t = \frac{v - v_0}{a} = \frac{0 - (20.83\ m/s)}{(-1.58\,\frac{m}{s^2})} = \boxed{13.18\ s}$$

(d) How far does the car travel before coming to rest?

$$\Delta x \ = \ x - x_0 \ = \ v_0 t + \frac{1}{2} a t^2$$

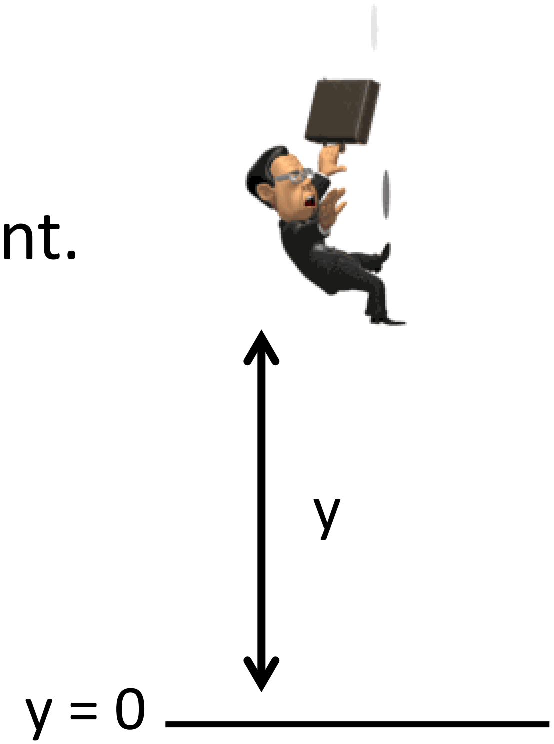
$$= \left(20.83\ \frac{m}{s}\right)(13.18\ s) + 2\left(-1.58\,\frac{m}{s^2}\right)(13.18\ s)^2$$

$= \boxed{137.31\ m}$ (about 450 ft)

Freely-falling objects

- The most important example of constant acceleration is an object in free fall.
- An object near the surface of Earth is falling freely under influence of gravity.
- **No forces** except gravity are acting.
- Then the acceleration will be constant.

  (see next slide)
- For vertical motion, we will use y to denote height (instead of x).
- Take origin at ground level (so y=0 at ground).



Freely-falling objects

- The acceleration due to gravity is **constant**.
- Because it is **downward**, it should be negative.

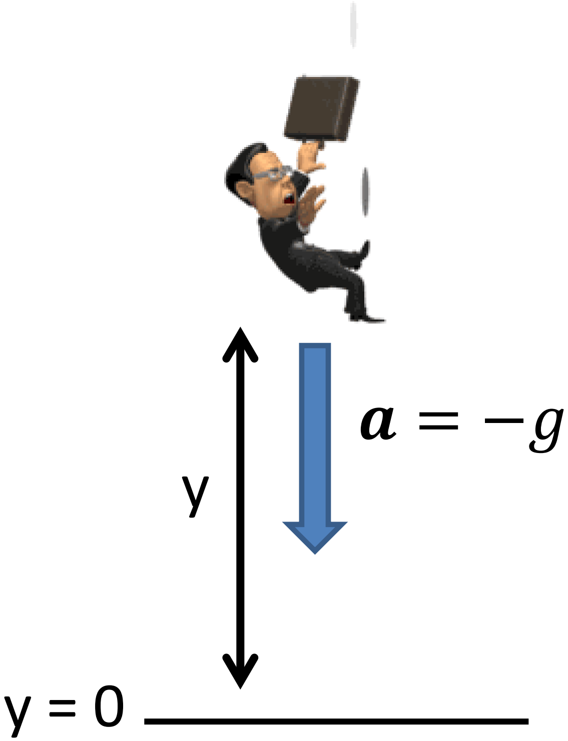
Gravitational acceleration:

$$\boldsymbol{a} = -g,$$

where:

$$g = 9.8\ \frac{m}{s^2} = 32\ \frac{ft}{s^2}$$

- **Warning:** $\boldsymbol{a}$ is constant and equal to this value **only** near the Earth’s surface.
- We’ll look at more general cases in a later chapter



Free fall

- Because gravitational acceleration is constant, we can use the constant acceleration equations.
- We just make the replacements:

$$a \rightarrow -g, \qquad x \rightarrow y$$

- Result:

$$\boxed{\begin{aligned} &1.\ \ y = y_0 + v_0 t - \frac{1}{2} g t^2 \\ &2.\ \ v = v_0 - gt \\ &3.\ \ v^2 = v_0^2 - 2g(y - y_0) \end{aligned}}$$

- Note: only **differences** in height matter. Can choose origin (y=0) anywhere you want, and measure differences from there.

Example 1

- The coyote is at the top of a cliff, 100 m high. He drops an Acme$^{\text{TM}}$ anvil, which starts from rest.

(a) What is the speed of the anvil at the bottom of the cliff?

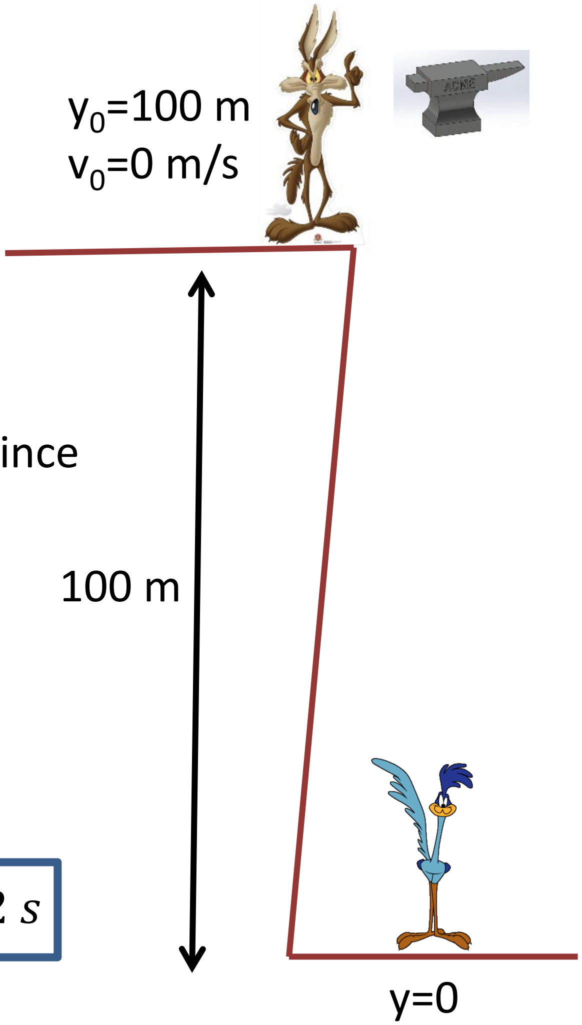
$$v = -\sqrt{v_0^2 - 2g(y - y_0)}$$

$$= -\sqrt{0 - 2\left(9.8\,\frac{m}{s^2}\right)(0 - 100m)} = \boxed{-44.3\,\frac{m}{s}}$$

← **Negative** root used, since v must be **downward**

(b) How long does it take to fall?

$$v = v_0 - gt \quad \rightarrow \quad t = \frac{v - v_0}{g} = \frac{0 - \left(-44.3\,\frac{m}{s}\right)}{9.8\,\frac{m}{s^2}} = \boxed{4.52\ s}$$



Note that we did **not** need to know the mass of the anvil. Objects of different mass fall at the same rate (ignoring air resistance).

Example 2

- Throw a ball straight upward with initial speed of 15 m/s. (Can take initial height to be $y_0 = 0.$)

(a) Find the height and velocity one second later.

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 = 0 + \left(15\ \frac{m}{s}\right)(1s) - \frac{1}{2}\left(9.8\ \frac{m}{s^2}\right)(1s)^2 = \boxed{10.1m}$$

$$v = v_0 - gt = \left(15\frac{m}{s}\right) - \left(9.8\ \frac{m}{s^2}\right)(1s) = \boxed{5.2\frac{m}{s}}$$

(b) Find the height and velocity after 2 seconds.

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 = 0 + \left(15\ \frac{m}{s}\right)(2s) - \frac{1}{2}\left(9.8\ \frac{m}{s^2}\right)(2s)^2 = \boxed{10.4m}$$

$$v = v_0 - gt = \left(15\frac{m}{s}\right) - \left(9.8\ \frac{m}{s^2}\right)(2s) = \boxed{-4.6\frac{m}{s}}$$

Example continued ⇨

Example 2 (continued)

(c) Find the maximum height. At what time is it reached?

To find maximum height, set $v = 0$ at that point.

$$v = v_0 - gt \quad \rightarrow \quad 0 = v_0 - gt \quad \rightarrow \quad \text{Solve for t.}$$

$$t = \frac{v_0}{g} = \frac{15\ m/s}{9.8\ m/s^2} = \boxed{1.53\ s}$$

$$\text{y} = y_0 + v_0 t - \frac{1}{2}gt^2 = 0 + \left(15\ \frac{m}{s}\right)(1.53s) - \frac{1}{2}\left(9.8\ \frac{m}{s^2}\right)(1.53s)^2$$
$$= \boxed{11.48\ m}$$

Consistency check: From parts a and b, we know the time of maximum height has to between 1s and 2s.

Example continued ⟹

Example 2 (continued)

(d) How long before the ball returns to initial height?

Now: $y = y_0 = 0$

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 \quad \rightarrow \quad 0 = 0 + v_0 t - \frac{1}{2} g t^2 = \left(v_0 - \frac{1}{2} g t \right) t$$

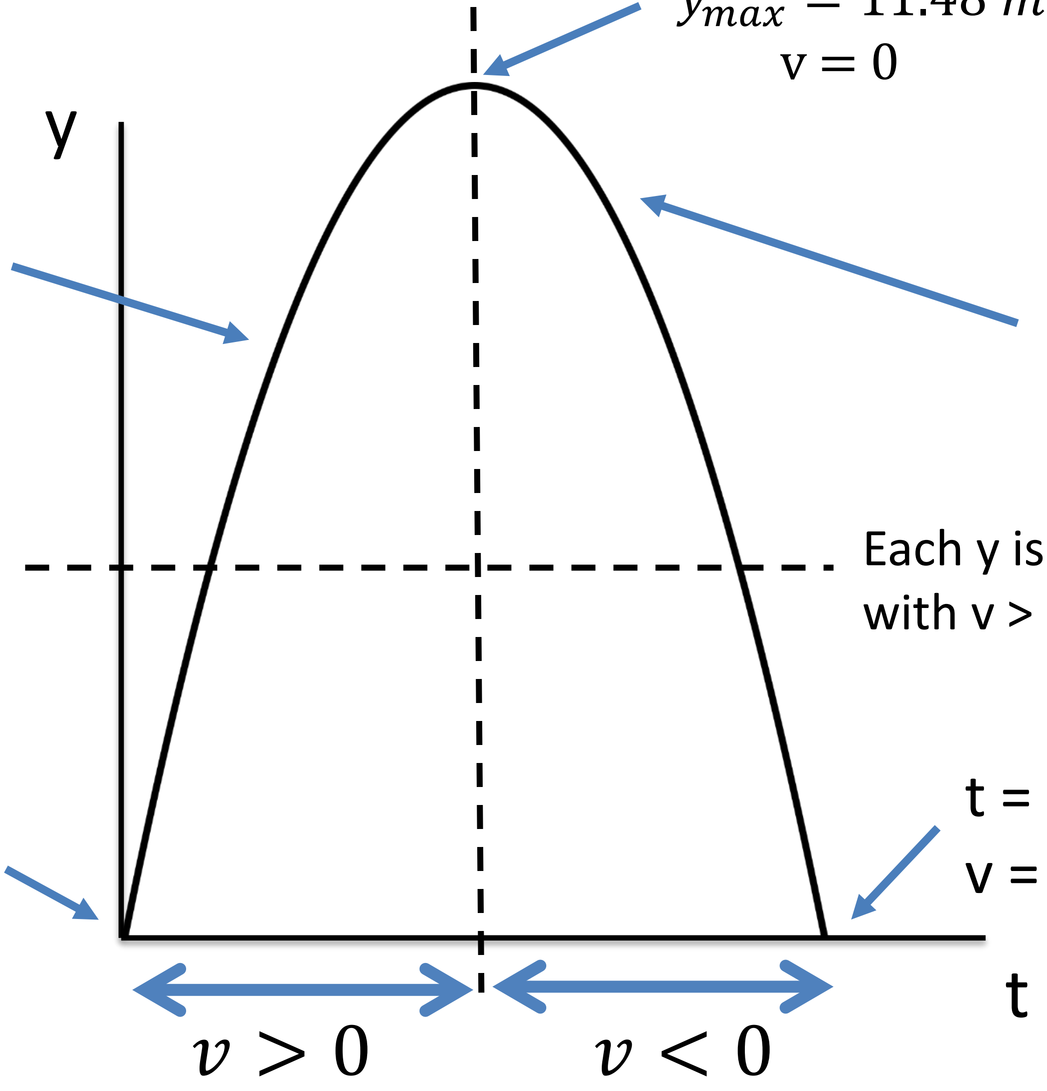
$$= 0 + \left(15 \ \frac{m}{s} \right) (1.53s) - \frac{1}{2} \left(9.8 \ \frac{m}{s^2} \right) (1.53s)^2$$

This has two solutions. One is t=0. The other:

$$t = \frac{2v_0}{g} = \frac{2(15 \frac{m}{s})}{9.8 \ \frac{m}{s^2}} = \boxed{3.06 \ s}$$

Example continued ⇒

Example 2 (continued)



$y_{max} = 11.48\ m$
v = 0

t = 1 s
y = 10.1 m
v > 0

t = 2 s
y = 10.4 m
v < 0

Each y is crossed twice, once with v > 0, once with v < 0

t = 0 s,
v = + 15 m/s

t = 3.06 s,
v = - 15 m/s

Example 3

- Krazy the Kaffienated Kat wants to catch a bee 2m above the ground. What must her initial launch speed be?

Know: $v = 0$ at max. height, $y = 2m$. Initially: $y_0 = 0$, $v_0 = ?$

We neither have nor want information about time, so use the equation that doesn't involve t.

$$v^2 = v_0^2 - 2g(y - y_0) \qquad \rightarrow \qquad 0 = v_0^2 - 2g(y - y_0)$$

Solve for v_0:

$$v_0 = +\sqrt{2g(y - y_0)} = +\sqrt{2(9.8\,\frac{m}{s^2})(2\,m)} = \boxed{6.26\,\frac{m}{s}}$$

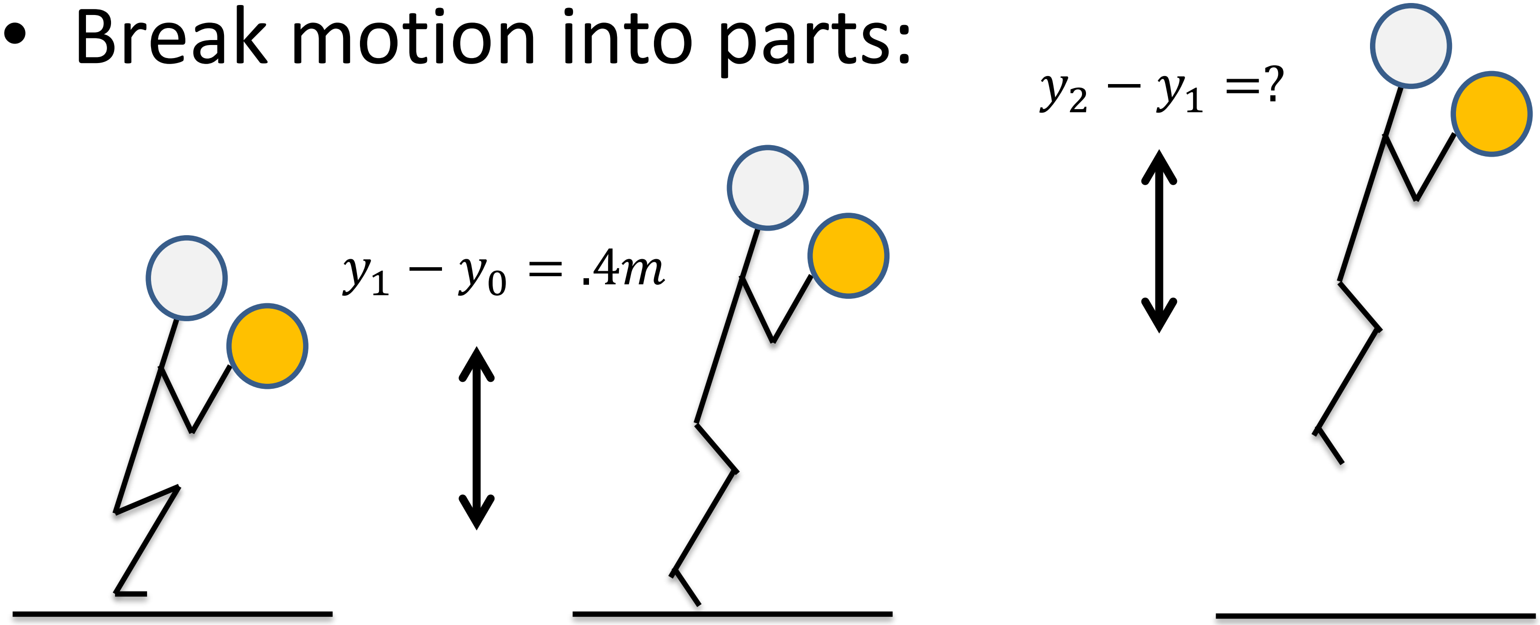
Use + root, since initial velocity must be upward

Example 4

- A basketball player is jumping to make a shot.
- His acceleration changes in the middle of his motion. We break his motion into two parts:
  - While in contact with the floor, the floor exerts a force against him: his acceleration is constant, but does not equal –g.
  - After leaving floor, gravity is now only force. So now a = -g
- Assume: While in contact with floor $a = +20\,\frac{m}{s^2}$. He accelerates through distance $y - y_0 = .4\ m$ while in contact with the floor.
- How high does he go after leaving the floor?

Example 4 (continued)

- Break motion into parts:



$y_1 - y_0 = .4m$

$y_2 - y_1 = ?$

At lowest point of crouch

$$t_0$$

$$y_0 = 0$$

$$v_0 = 0$$

At minimum height

Just leaving ground

$$t_1$$

$$y_1 = .4\ m$$

$$v_1 = ?$$

At maximum height

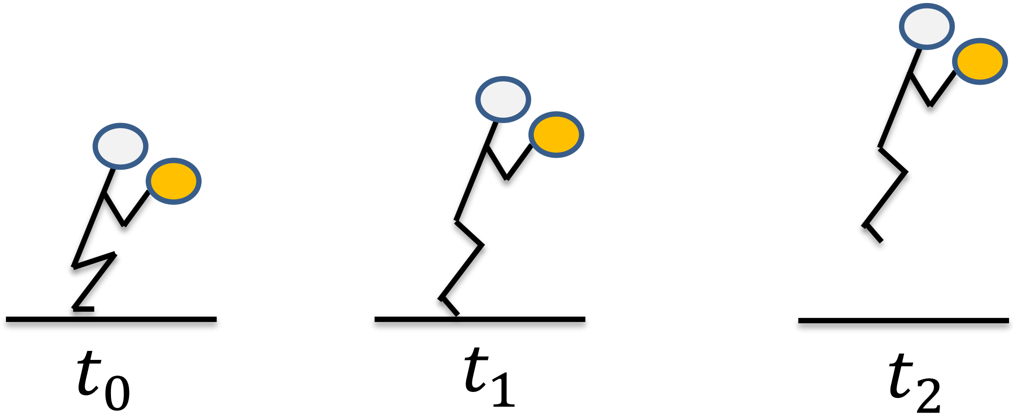
$$t_2$$

$$y_2 = ?$$

$$v_2 = 0$$

At maximum height

Example 4 (continued)



- From t_0 to t_1:

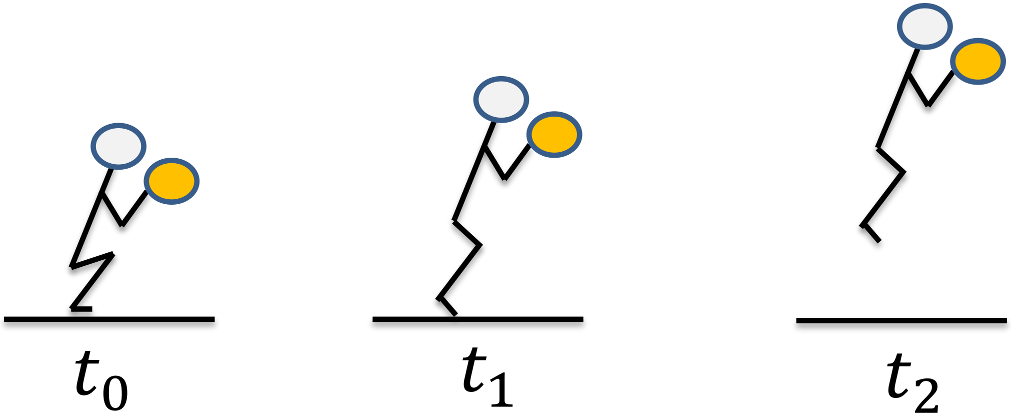
$$v^2 = v_0^2 + 2a(y - y_0)$$

$$v_1 = +\sqrt{v_0^2 + 2a(y_1 - y_0)} = +\sqrt{0 + 2(20\,\frac{m}{s^2})(.4\,m)} = \boxed{4.0\,\frac{m}{s}}$$

Upward

This is the velocity as he leaves the ground. Now look at the second part of the motion, from t_1 to t_2

Example 4 (continued)



- From t_1 to t_2:

Start same was as before, but now:

At maximum height

$$a = -g \,, \qquad \text{initial speed is } v_1 = 4.0\,\frac{m}{s}, \quad \text{and} \quad v_1 = 0$$

So: $v_2^2 = v_1^2 - 2g(y_2 - y_1)$

$$y_2 - y_1 = \frac{v_2^2 - v_1^2}{-2g} = \frac{0 - \left(4.0\,\frac{m}{s}\right)^2}{-2\left(9.8\,\frac{m}{s^2}\right)} = \boxed{.82\,m}$$

(about 2.68 ft)

Ch. 20 – The 2nd Law of Thermodynamics

Laws of Thermodynamics

- Zeroth and first laws of thermodynamics:
  - **0th law:** Two if objects A and B are both in thermal equilibrium with object C, then they are in equilibrium with each other.

    (This is a technical condition needed to define thermal equilibrium.)
  - **1st law:** $Q = W + \Delta E$

The first law expresses conservation of energy.

The second law describes the ***usefulness*** of the available energy, and what can and can't be done with it.

The 2nd law is intrinsically statistical in nature.

Second Law

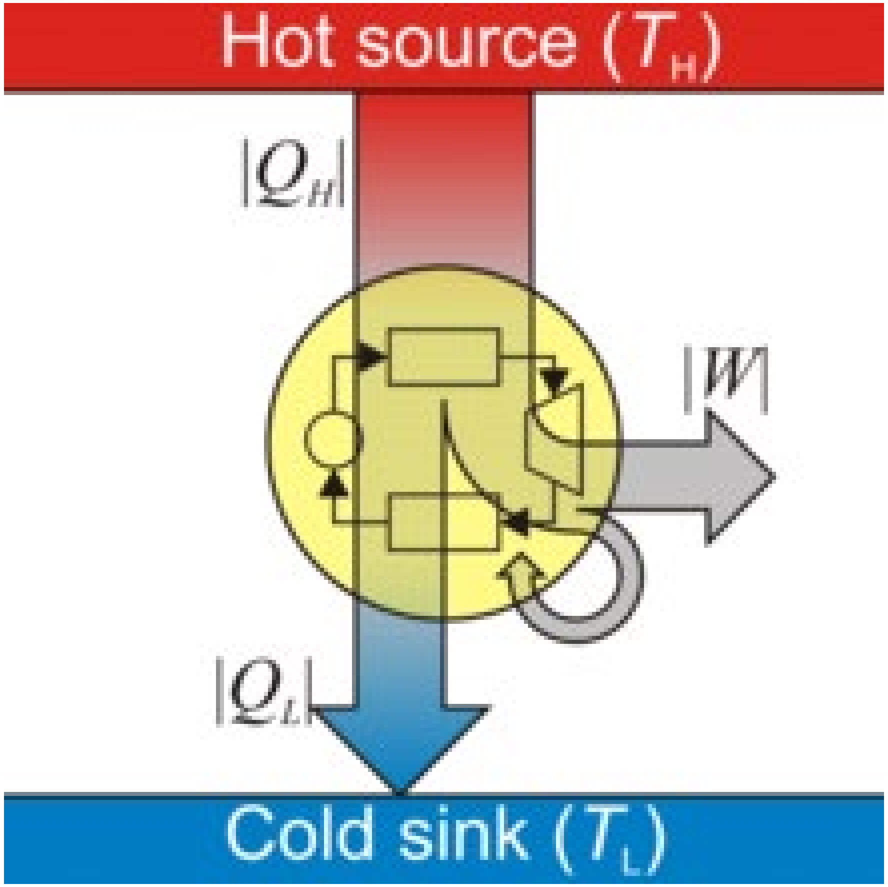
There are many equivalent ways to state the 2nd law:

- **Clausius Statement:** Heat flows from hot to cold objects. To force heat in the other direction requires the input of additional energy.
- **Kelvin-Planck Statement:** No device is possible whose sole effect is transformation of heat into work. Some unusable waste heat will always be left over.
- **Carnot Statement:** No engine can be of higher efficiency than a Carnot engine operating at the same temperatures.
- **Entropy Statement:** The entropy of a closed system will always increase in any realistic process.

We'll flesh out what all of these statements mean in this chapter.

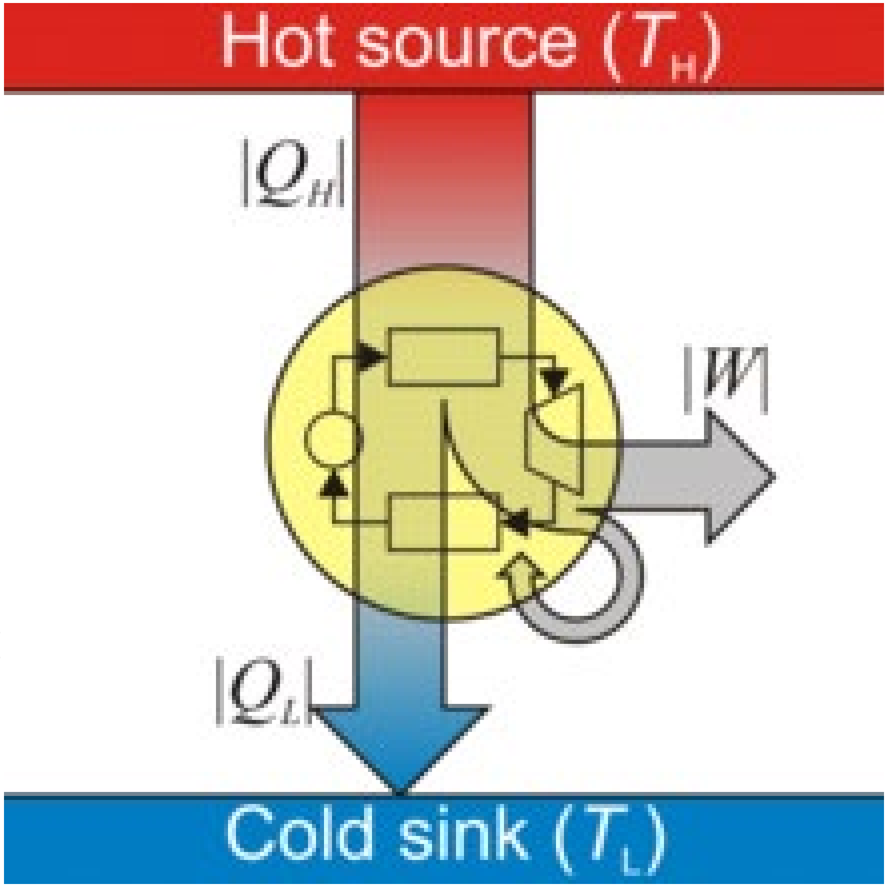
Heat engines

- Heat engines are a schematic version of a real engine.
- To be as general as possible, only the essential aspects are included.
- Every heat engine works by converting heat into work.
- Examples of heat engines:
  - Gasoline engines
  - Chemical reactions over a flame
  - Muscles
  - Individual cells



Heat engines

- A hot object (the **high-temperature reservoir**) at T_H serves as a source of heat, Q_H.
- Q_H is the energy that drives the engine. A portion of it is converted into work.
- The left-over energy not used for work must be expelled to something of lower temperature.
- This is the **low-temp. reservoir**, at temp. $T_{L.}$ Heat Q_L is expelled to it.
- Signs: $Q_H > 0$, Q_L<0.



Heat Engines

- Energy conservation (1^{st} law) tells us: $\boxed{Q_H = Q_L + W}$
- The second law tells us that the waste heat Q_L can't be zero:

$$Q_L \neq 0 \quad \rightarrow \quad Q_H > W$$

- So more heat has to be put in than we get back out as useful work.
- In other words the efficiency can't be 100%.
- It will turn out that the maximum possible efficiency will depend on the operating temperatures T_L and T_H.

Efficiency of a Heat Engine

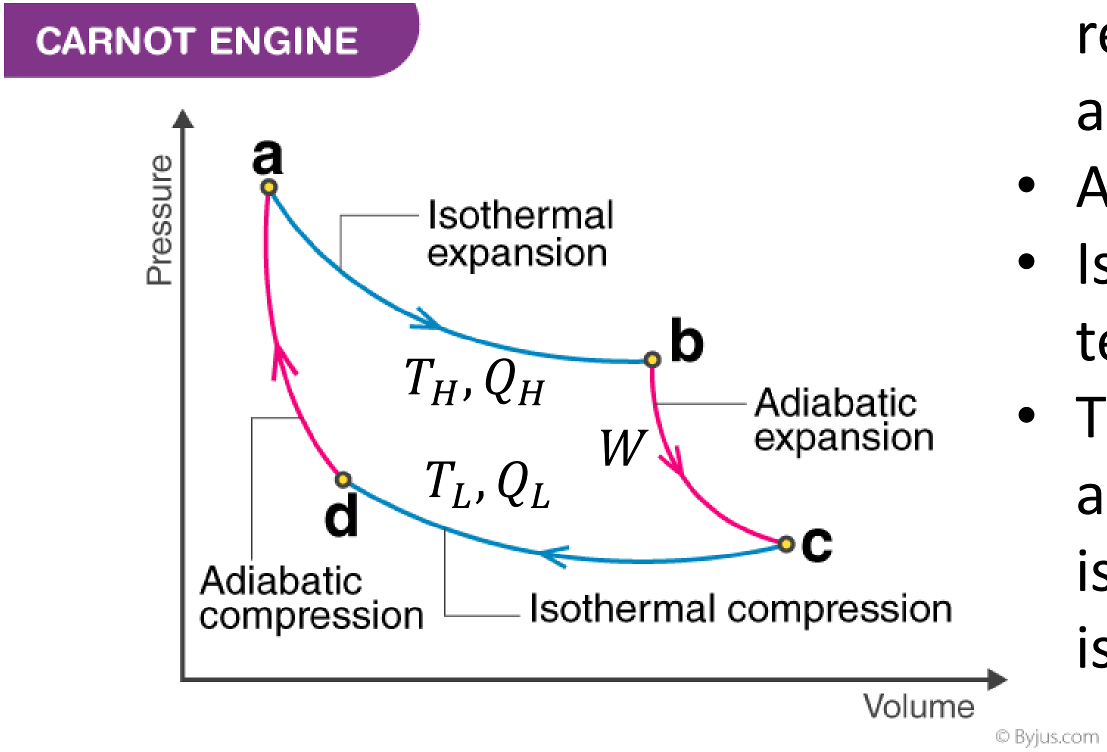
- The **efficiency** e is defined by: $\boxed{e = \frac{W}{Q_H}}$
- Often is written as a percentage.
- Using the first law, we can put this in another useful form:

$$e = \frac{Q_H - Q_L}{Q_H} \quad \rightarrow \quad \boxed{e = 1 - \frac{Q_L}{Q_H}}$$

- Since $\frac{Q_L}{Q_H} > 0$, this again implies $e < 1$.
- In fact the limit is more stringent than this. To see that, let’s ask what the most efficient engine possible looks like.

Carnot Engine

- Sadi Carnot (1824) showed that the following process has the maximum possible efficiency at given values of T_L and T_H.


CARNOT ENGINE
Pressure
a
Isothermal expansion
b
T_H, Q_H
Adiabatic expansion
W
T_L, Q_L
d
c
Adiabatic compression
Isothermal compression
Volume
© Byjus.com

- This process is **cyclic**: it returns to its starting point and repeats.
- Adiabatic means no heat.
- Isothermal means constant temperature.
- The work is done during the adiabatic portions, the heat is transferred during the isothermal parts.

Carnot Efficiency

- The Carnot cycle is unique because it has the property that the heats and temperatures obey:

$$\frac{Q_L}{Q_H} = \frac{T_L}{T_H}$$

- Plugging this into the definition of efficiency, we get the efficiency of the Carnot engine:

$$\boxed{e_{Carnot} = 1 - \frac{T_L}{T_H}}$$

- The Carnot statement of 2nd law is that at given T_L and T_H, the efficiency e of any real engine is less than this:

$$\boxed{0 < e < e_{Carnot} < 1}$$

Example 1

- A gasoline engine takes in 2500 J of heat and delivers 500 J of work per cycle. The heat of combustion of gasoline is $L_C = 5 \times 10^7 \frac{J}{kg}$.

(a) What is the efficiency of the engine?

$$e = \frac{W}{Q_H} = \frac{500\,J}{2500\,J} = .2 = \boxed{20\%}$$

(b) How much waste heat is discarded per cycle?

$$Q_L = Q_H - W = 2500 - 500 = \boxed{2000\,J}$$

Example 1 (continued)

(c) How much gas is burned per cycle?

$$m = \frac{Q_H}{L_c} = \frac{2500}{5 \times 10^7} = 5 \times 10^{-5} kg = \boxed{.05\ g}$$

(d) If the engine goes through 100 cycles per second, what is the total power output of the engine?

$$P = \frac{work}{time} = \left(\frac{work}{cycle}\right)\left(\frac{cycles}{time}\right)$$

$$= \left(500 \frac{J}{cycle}\right)\left(100\ \frac{cycles}{s}\right) = \boxed{50{,}000\ W}$$

(= 67 hp, since 1 hp=746 W)

Example 2

- A Carnot engine takes in 2000 J of heat from reservoir at 500 K, does some work, and discards some heat to a reservoir at 2350 K. Find W, Q_L, and e.

$$Carnot \rightarrow \boldsymbol{\frac{Q_L}{Q_H} = \frac{T_L}{T_H}} \rightarrow \quad Q_L = Q_H \frac{T_L}{T_H} = (2000)\left(\frac{350}{500}\right) = \boxed{1400\ J}$$

$$W = Q_H - Q_L = 2000 - 1400 = \boxed{600J}$$

$$e = 1 - \frac{Q_L}{Q_H} = 1 - \frac{1400}{2000} = .3 = \boxed{30\%}$$

-- OR --

$$\boldsymbol{e = 1 - \frac{T_L}{T_H}} = 1 - \frac{350}{500} = .3 = \boxed{30\%}$$

Remember: the equations in red are true **only** for Carnot engines

Entropy

- Suppose we add heat to an object at temperature T.
- Then the **entropy change** of the object is defined to be:

$$\boxed{\Delta S = \int \frac{dQ}{T}}$$

(in units of J/K)

- Note that in general the temperature will be changing during the process, and the heat on top may be a function of T.
- Special case: if T is constant, then the integral is not needed and we have:

$$\boxed{\Delta S = \frac{Q}{T}}$$

(Constant temp. ***only***)

Entropy

- Note:
  - Only changes in entropy matter, not the actual value.
  - If heat is added, $\Delta S > 0$. If heat is removed, $\Delta S < 0$.

- It the temperature is changing, you need to use the specific heat:

$$Q = mC\Delta T \quad \rightarrow \quad dQ = mC\, dT$$

What does entropy mean?

- Roughly, entropy is a measure of disorder.
- A gas is much more disordered than a solid, and so will have higher entropy.
- Any material will increase in entropy as temperature increases, since there will be more random thermal motion.
- Be careful though: entropy is a measure of disorder at the ***microscopic*** level only.
  - For example, a messy room does not necessarily have more entropy than a well-ordered room.

What does entropy mean?

- Another way to think of entropy is in terms of information and probability.
- States that can be formed in many ways have more probability to occur and have higher entropy than states that are harder to form and less probable.
- Entropy: $S =$ information required to specify state
- It requires less information to describe a highly ordered state than a disordered one: so disordered states have higher entropy.
- The information-based interpretation of entropy is especially useful in engineering.

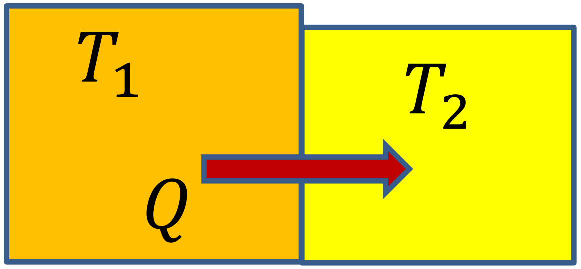
Entropy and the 2nd Law

Entropy version of the second law: The entropy **of a closed** system can only stay the same or increase. It can never decrease.

- **Important:** Notice that this ***only*** applies to a closed system, one that exchanges no energy with its surroundings.
- More compactly: for a closed system, $\Delta S \geq 0$.

Entropy and the 2^{nd} law

- Consider, for example, two objects of different temperatures in contact and exchanging heat with each other.



- Assume $T_1 > T_2$.
- Then $Q_1 < 0$ and $Q_2 > 0$.
- Take Q to be the absolute value: $Q_2 = -Q_1 = Q$.
- So:

$$\Delta S_1 = \frac{Q_1}{T_1} < 0\,, \qquad \Delta S_2 = \frac{Q_2}{T_2} > 0$$

- But since $T_1 > T_2$, it follows that $|\Delta S_1| < |\Delta S_2|$.
- Total: $\Delta S_1 + \Delta S_2 > 0$, in accordance with 2^{nd} law.

Smaller & negative

Larger & positive

Entropy and the 2^{nd} law

Another way to think of entropy and the second law:

- When energy is exchanged between objects it becomes *degraded* (less useful), in the sense that it becomes spread out over multiple objects and is lowered in temperature.
- To do more work, you have to find an even lower-temperature object for this heat to flow to, which becomes harder with each heat exchange.
- So heat at high temperature is easier to use for work than heat at low temperature.
- The entropy change is a measure of the degradation of the energy at each step.

Carnot engines and the 2nd law

- Now we can explain why no engine can be more efficient than the Carnot engine.
- Entropy changes of reservoirs: $\Delta S_H < 0, \Delta S_L < 0$
- Second law requires $\left|\frac{\Delta S_L}{\Delta S_H}\right| \geq 1$ (so that $\Delta S_{total} \geq 0$)
- So: $\frac{Q_L}{T_L} \cdot \frac{T_H}{Q_H} \geq 1 \;\; \rightarrow \;\; \frac{Q_L}{Q_H} \geq \frac{T_L}{T_H}$
- Therefore:

$$e = 1 - \frac{Q_L}{Q_H} \leq 1 - \frac{T_L}{T_H} = e_{carnot}$$

- The Carnot engine is right on the boundary of all of these inequalities. All real engines lie on one side of it and all engines on the other side would have negative ΔS, violating the second law.

Example 3

- 1 kg of ice at 0°C is melted, with no temperature change. Compute the entropy change.

- Since the temperature is constant, we don't need the integral:

$$\Delta S = \frac{Q}{T} = \frac{mL_F}{T} = \frac{(1\ kg)\left(3.34 \times 10^5\,\frac{J}{kg}\right)}{273\ K}$$

$$= \boxed{1223\ J/K}$$

ΔS is positive because the ice is **gaining** energy. The surroundings will have change that is smaller, but negative.

Example 4

- 0.1 kg of steam at 100°C condenses in a cold pan. Find ΔS, again assuming no temperature change.

$$\Delta S = \frac{Q}{T} = \frac{-mL_V}{T} = \frac{-(.1\ kg)\left(22.6 \times 10^5 \frac{J}{kg}\right)}{373\ K}$$

$$= -6.06 \times 10^2 \frac{J}{K} = \boxed{-606 \frac{J}{K}}$$

Again, this can be negative because the steam is a closed system. It is exchanging heat with its surroundings. The surrounding environment will have a larger positive entropy change, so that the total ΔS will be positive.

Example 5

- A Carnot engine has $T_H = 500\ K, T_L = 350$, and $Q_H = 2000\ J$. What is ΔS of engine per cycle?
- During the adiabatic portions 1 and 3:

$$Q = 0 \quad \rightarrow \quad \Delta S_1 = \Delta S_3 = \frac{Q}{T} = 0$$

- During isothermal process 2:

$$\Delta S_2 = \frac{Q_H}{T_H} = \frac{2000\ J}{500\ K} = 4\ J/K$$

- During isothermal process 4:

$$Q_4 = -Q_L = -Q_H\left(\frac{T_L}{T_H}\right) = -(2000J)\left(\frac{350\ K}{500\ K}\right) = \ -1400\ J$$

$$\Delta S_4 = \frac{Q_4}{T_L} = \frac{-1400\ J}{350\ K} = -4\frac{J}{K} \quad \rightarrow \quad \boxed{\Delta S_{total} = 0}$$

This verifies that Carnot engines provide a border between positive and negative entropy engines as stated earlier.

Example 6

- 1 kg of water goes from 0℃ to 100℃, remaining liquid the entire time. Find ΔS.
- This time, temperature is changing so we need the integral version.

$$\Delta S = \int \frac{dQ}{T} = \int_{T_0}^{T} \frac{mC\,dt}{T} = mC \int_{273}^{373} \frac{dt}{T}$$

$$= mC \ln \frac{373K}{273K} = \boxed{1305\,J/K}$$

Example 7

- 2 moles of gas expand isothermally from 40 L to 80 L at 273 K. Find ΔS.
- Converting volumes: $V_0 = .04\, m^3, V = .08\, m^3$
- Isothermal: $\Delta E = 0$ (since $\Delta E = mc\Delta T$ and $\Delta T = 0$)
- So:

$$Q = \Delta E + \mathrm{W} = \mathrm{W} = \int P\, dV = \int_{V_0}^{V} \frac{nRT}{V} dV$$

$$= nRT \int_{V_0}^{V} \frac{1}{V} dV = nRT\ \ln \frac{V}{V_0} = (2)(8.31)(273) \ln \frac{.08}{.04}$$

$$= 3147\, J$$

$$\Delta S = \frac{Q}{T} = \frac{3147}{273} = \boxed{11.52\ J/K}$$

Ch. 3: Vectors and Two-Dimensional Motion

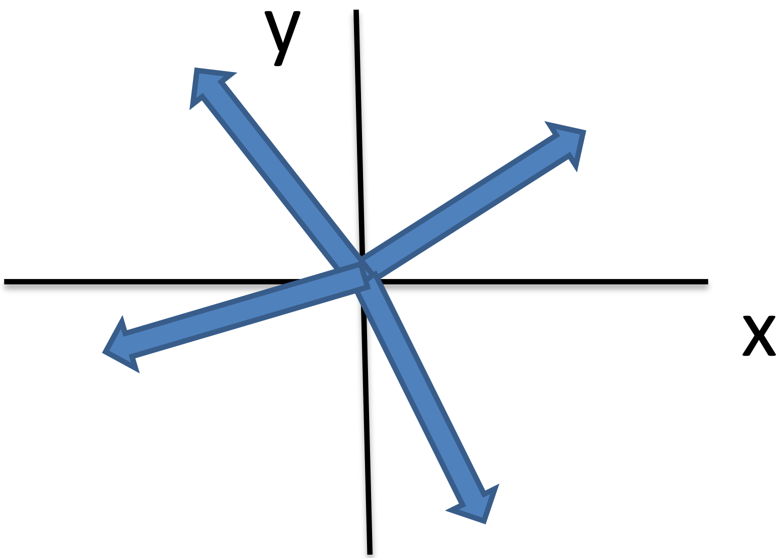
One Dimension

- So far: we have been dealing with motion in one dimension (along a single line)
- In one dimension, we can describe each quantity (position, velocity, acceleration, etc.) by a single number.
- Directions are given by a + or – sign.

v = + 5 m/s **v = + 5 m/s**

Two Dimensions

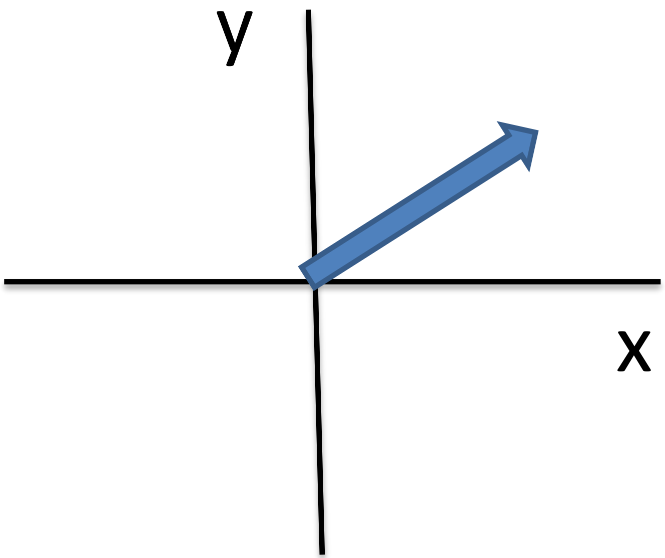
- In two dimensions, we have two coordinates (x = horizontal and y = vertical)
- There are now an **infinite number** of possible directions for movement.
- A single plus/minus sign will no longer cut it.



A few of the possible directions an object can move in two dimensions

Two Dimensions

- In **two dimensions**, quantities describing motion (velocity, acceleration, etc.) require **two numbers** to describe them.
- Similarly, in **three dimensions** each quantity is described by **three numbers**, etc.



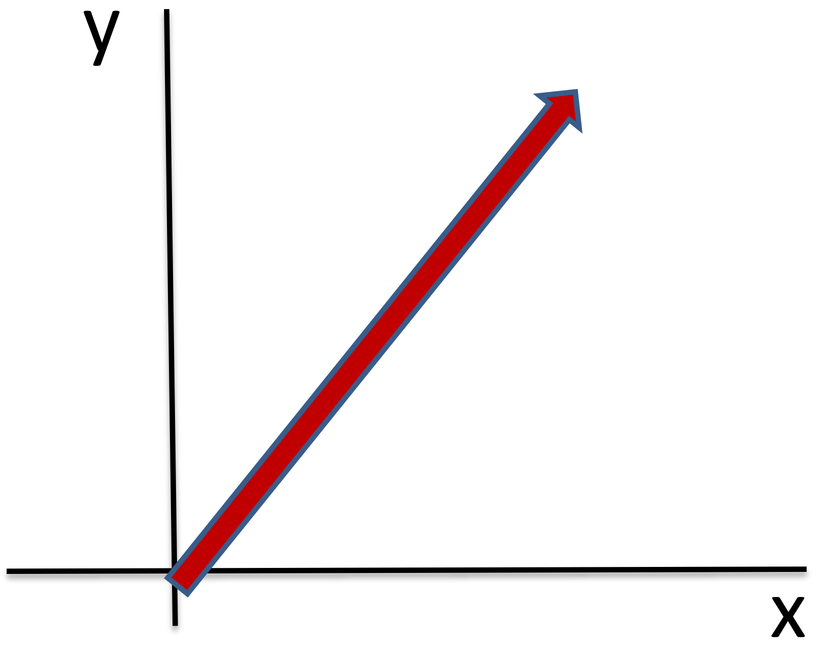
- This extra information is needed to describe the direction of motion.
- Quantities that encode information are called **vectors.**
- Quantities with no direction are called **scalars**.

Vectors versus scalars

- A **scalar** is an object that has just a size (magnitude). It is just a number
- A **vector** is an object with size **and** direction. Think of vectors as arrows.
- **Examples:**
  - **Scalars:** temperature, pressure, charge, energy
  - **Vectors:** velocity, acceleration, force, electric and magnetic fields.

Vectors (2D)

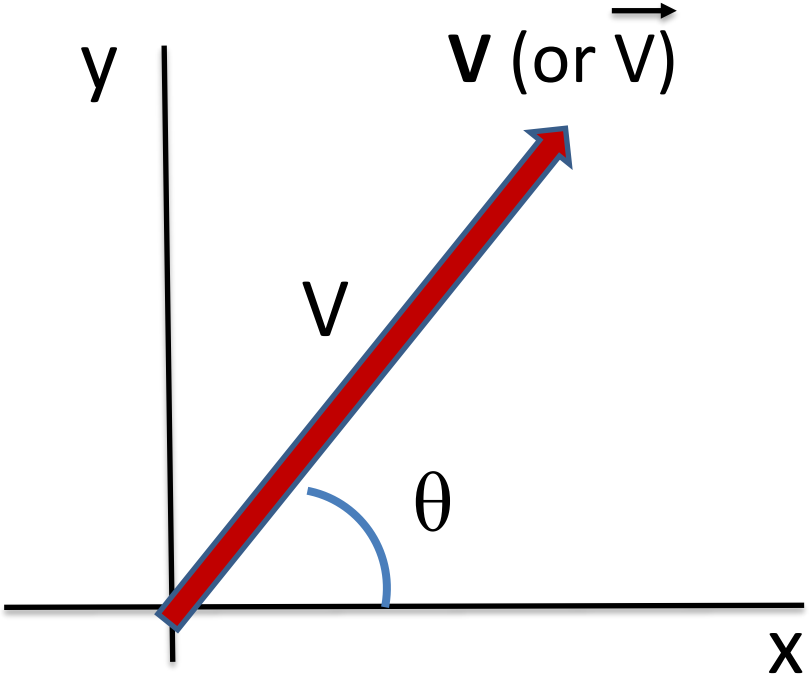
- Think of vectors as arrows. They carry two pieces of information:
    - **Magnitude or size** (length of the arrow)
    - **Direction** (angle of arrow from positive x-axis)



- These two numbers provide a complete description of a vector (in two dimensions)
- We'll soon see a second equivalent (and more useful) description.

Vectors: Notation

- Vectors are given names that are boldface or have arrows over them, to distinguish them from scalars. (**V**, $\vec{V}$, **A**, $\vec{A}$, $\boldsymbol{\alpha}$, $\vec{\alpha}$,……)



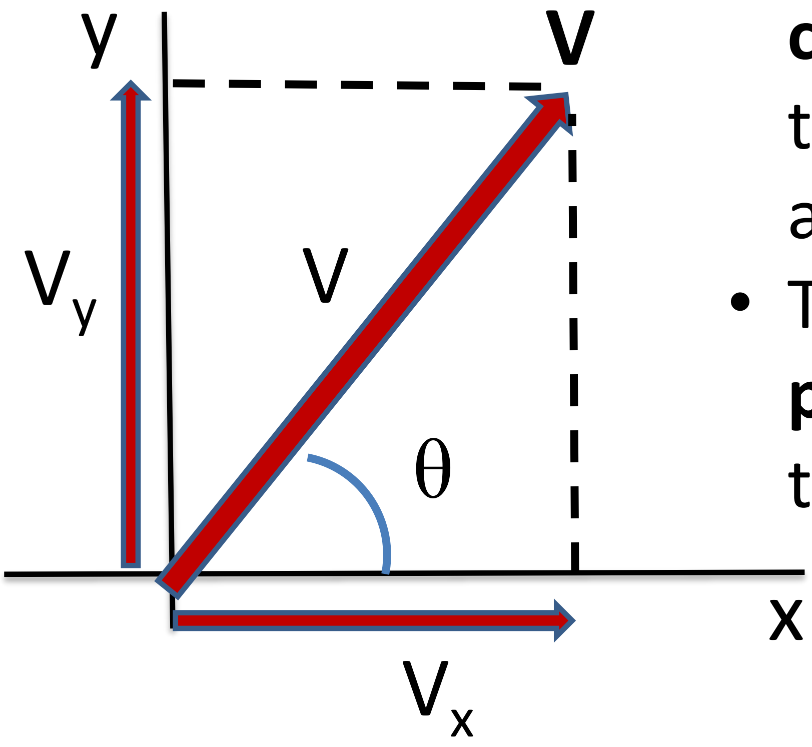
- The magnitude is denoted by the same name, but without the bold or the arrow.
- The angle is usually called θ or φ.

Theta Phi

- So one vector is equivalent to two numbers: **V**= (V, θ)
- (In n dimensions, one vector is equivalent to n numbers)

Vectors: Components

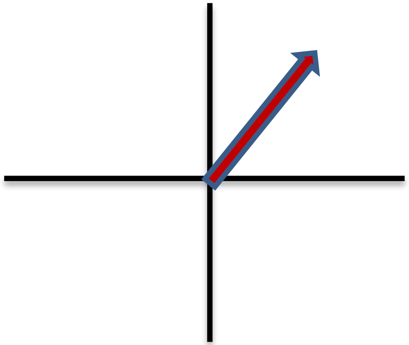
- There is a second way to describe a vector using two different numbers, called **components.**



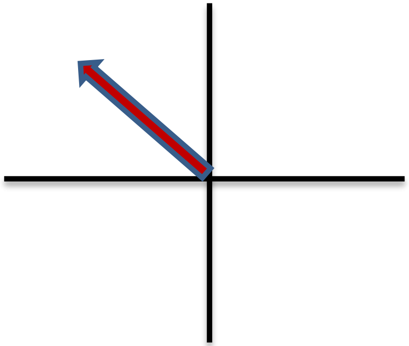
- The **x-component** V_x and the **y-component** V_y are the distances the vector extends horizontally and vertically.
- They are the shadows or **projections** of the vector onto the two axes.

Signs

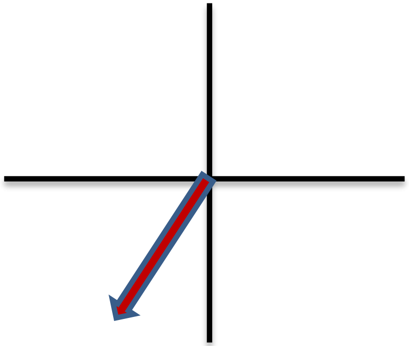
- Note that the **magnitude** of a vector **must always be positive**.
- But the **components** can be positive or negative, depending on direction.
- $V_x > 0$ → points right, $V_x < 0$ → points left
- $V_y > 0$ → points up, $V_y < 0$ → points down



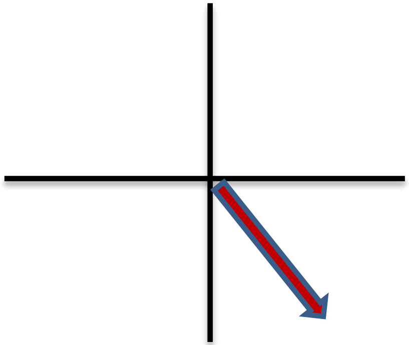
$V_x>0$
$V_y>0$



$V_x<0$
$V_y>0$



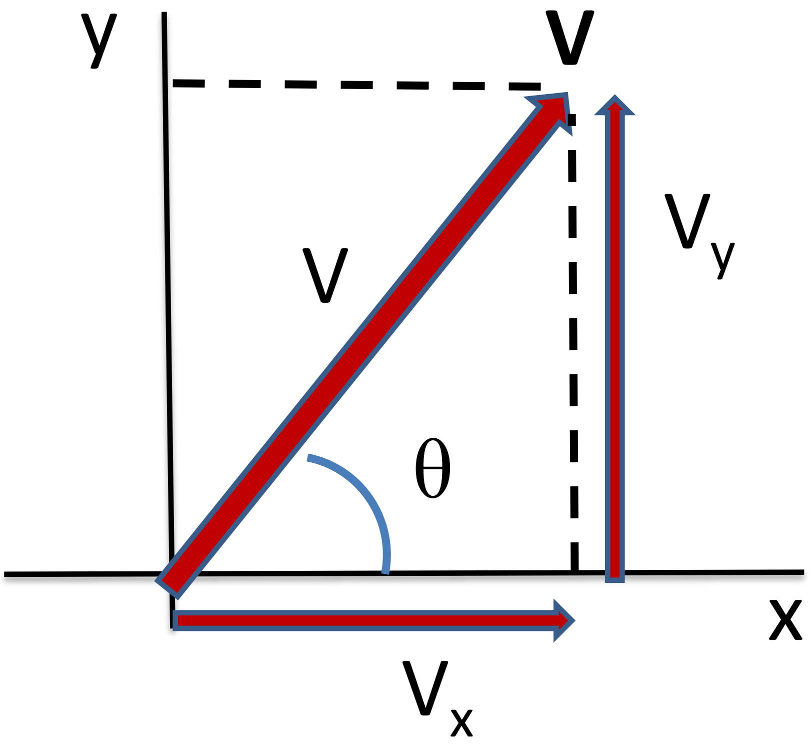
$V_x<0$
$V_y<0$



$V_x>0$
$V_y<0$

Converting between two descriptions

- We have two descriptions of the vector: in terms of magnitude and angle, and in terms of x- and y-components.



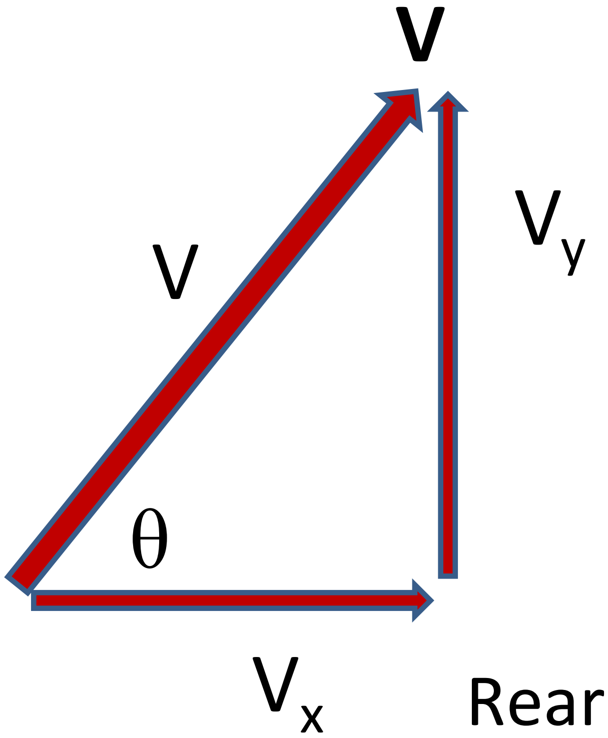
- To translate between them, notice that magnitude forms the hypotenuse of a right triangle.
- The two components form the two sides.

Converting between two descriptions

- From the Pythagorean theorem:

$$V^2 = V_x^2 + V_y^2$$

- From the definitions of the trig functions:



$$\sin\theta = \frac{opp}{hyp} = \frac{V_y}{V}$$

$$\cos\theta = \frac{adj}{hyp} = \frac{V_x}{V}$$

$$\tan\theta = \frac{opp}{adj} = \frac{V_y}{V_x}$$

Rearrange these to get what we need

Converting between two descriptions

- Magnitude and angle from components:

$$\boxed{V = \sqrt{V_x^2 + V_y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{V_y}{V_x}}$$

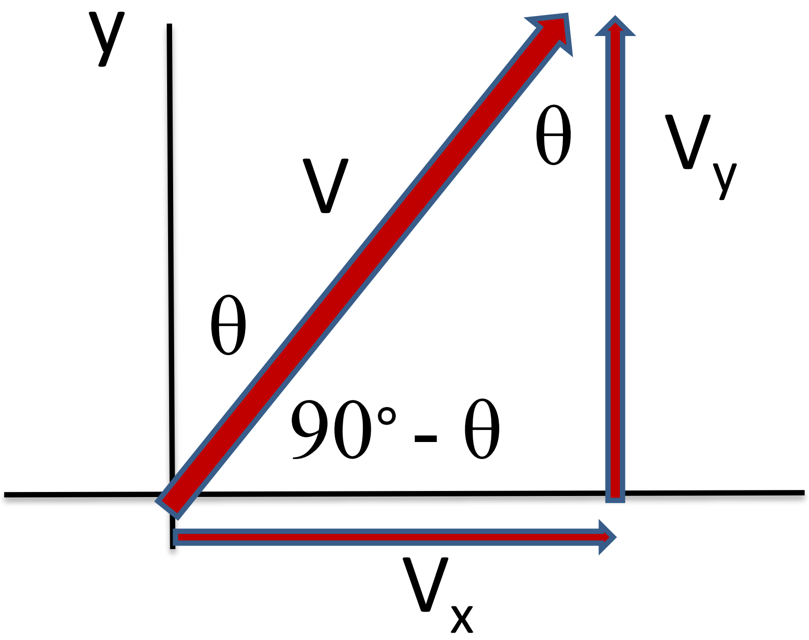
- From the definitions of the trig functions:

$$\boxed{V_x = V\cos\theta \quad \text{and} \quad V_y = V\sin\theta}$$

- These are the basic equations for working with vectors.

Converting between two descriptions

- **Be careful:** the previous equations assume is the angle from the **x-axis**.
- Often in problems you will be given the angle from the **y-axis**. In that case, sines and cosines get interchanged:

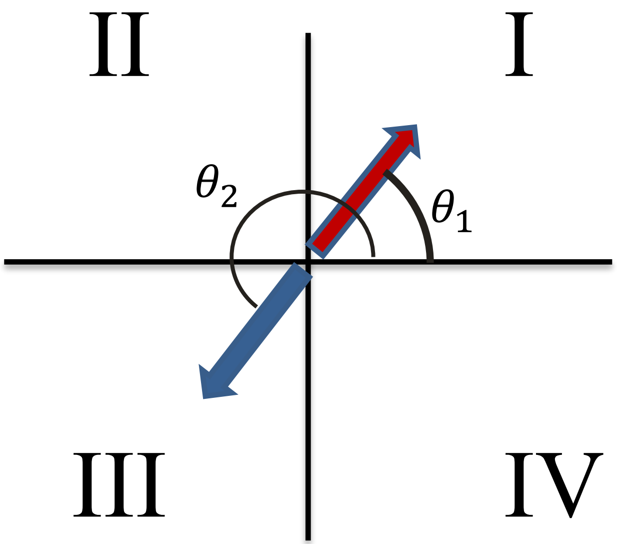


$$V_x = V\cos(90° - \theta)$$
$$= V\sin(\theta)$$

$$V_y = V\sin(90° - \theta)$$
$$= V\cos(\theta)$$

Converting between two descriptions

- **One more warning**: The inverse tangent is double-valued: check quadrant to make sure your calculator is giving the value you need.
- If it is in the wrong quadrant, add 180°.



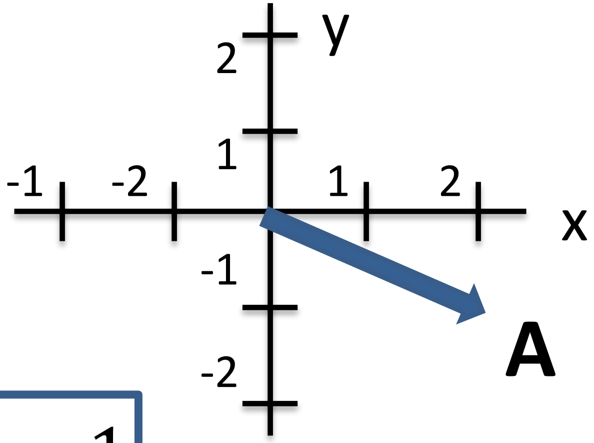
The angles of the two vectors shown (180° apart, in quadrants I and III) have the same inverse tangent:

$$\tan\theta_1 = \frac{V_{1y}}{V_{1x}} = \frac{(-V_{2y})}{(-V_{2x})} = \tan\theta_2$$

$$(\theta_2 = \theta_1 + 180°)$$

Example 1

- A vector **A** is as shown.



(a) Find the components.

These can be read off directly from the picture:

$$\boxed{A_x = +2, \quad A_y = -1}$$

(b) Find the magnitude and direction

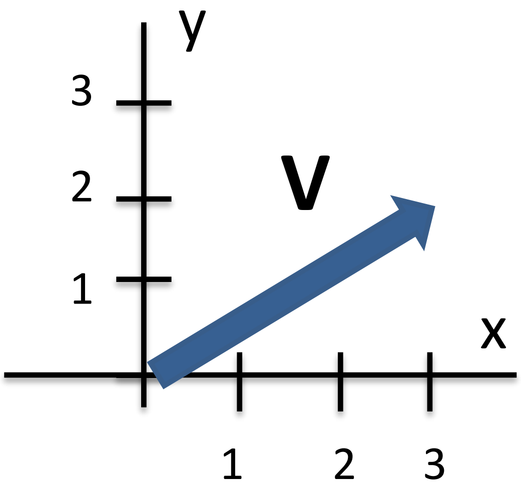
$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{(2)^2 + (-1)^2} = \boxed{\sqrt{5}} \approx 2.236$$

$$\theta = \tan^{-1}\frac{A_y}{A_x} = \tan^{-1}\left(\frac{-1}{2}\right) = \boxed{-26.6^\circ} = 333.4^\circ$$

$$= 5.82\, rad$$

Example 2

- A hiker walks three miles east, then 2 miles north. What is the magnitude and direction of the total displacement?



Given: $V_x = 3\,miles, V_y = 2\,miles$

$$V = \sqrt{V_x^2 + V_y^2} = \sqrt{3^2 + 2^2}$$

$$= \sqrt{13} = \boxed{3.61\,miles}$$

$$\theta = \tan^{-1}\frac{V_y}{V_x} = \tan^{-1}\frac{2}{3} = \boxed{33.69°}$$

$(= .59\,rad,$ N of E)

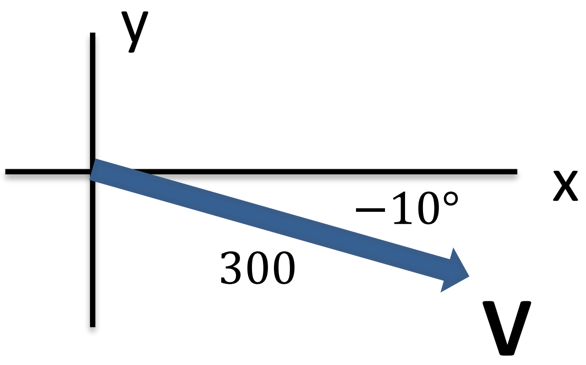
Example 3

- An airplane is moving at a constant speed of 300 mph, at 10° below horizontal. Find the components of the velocity.

Given: $V = 300\,mph, \quad \theta = -10°$

$$V_x = V\cos\theta = 300\ \cos(-10°) = \boxed{295.44\,mph}$$

$$V_y = V\sin\theta = 300\ \sin(-10°) = \boxed{-52.09\,mph}$$

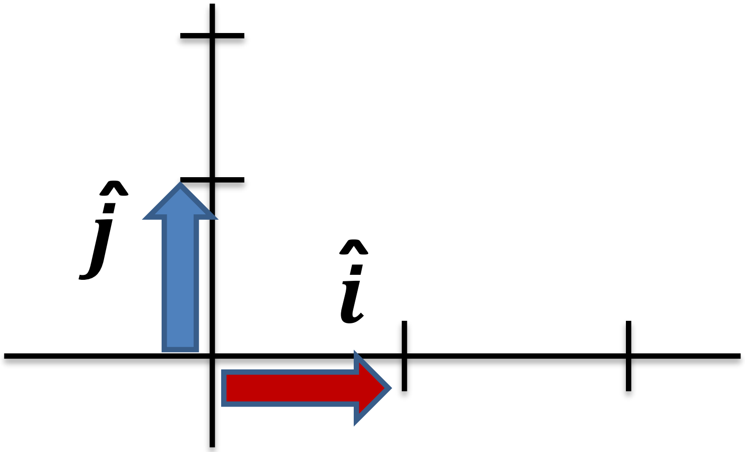


Check answer:

$$V = \sqrt{V_x^2 + V_y^2} = \sqrt{295.44^2 + (-52.09)^2} = 300\,mph$$

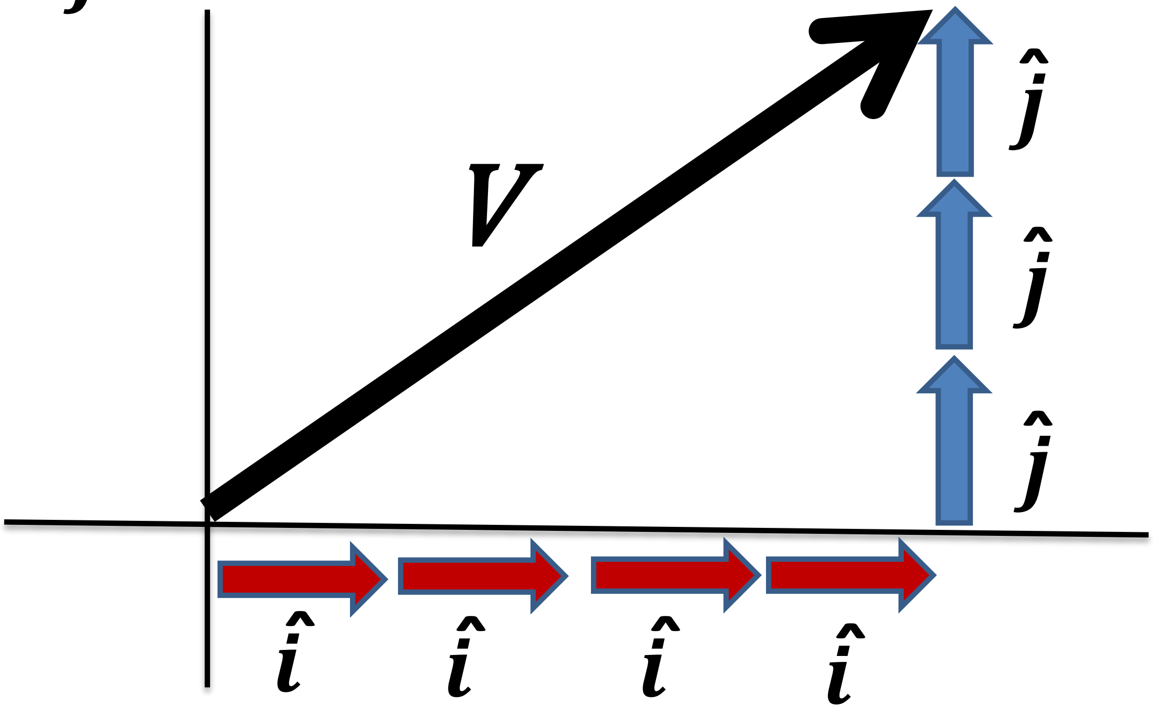
Unit Vector Notation

- When we start adding vectors, we will need a better notation for them.
- Define **unit vectors**: $\hat{\boldsymbol{i}}, \quad \hat{\boldsymbol{j}}$
- These are vectors of length 1, pointing along the axes. The hats will always indicate unit vectors.


$\hat{\boldsymbol{j}}$
$\hat{\boldsymbol{i}}$

Unit Vector Notation

- Any vector can be built out of unit vectors.
- For example, the vector below can be written as $V = 4\,\hat{\imath} + 3\hat{\jmath}$


V
$\hat{\jmath}$
$\hat{\jmath}$
$\hat{\jmath}$
$\hat{\imath}$
$\hat{\imath}$
$\hat{\imath}$
$\hat{\imath}$

Vectors in three dimensions

- In three dimensions, there are three coordinates (x, y, z) and three components for each vector ( )
- There are also now three unit vectors: $\hat{\boldsymbol{\imath}}, \; \hat{\boldsymbol{\jmath}}, \; \hat{\boldsymbol{k}}$
- There will be **two** angles (one from the z-axis, one from the xy-plane).
- The Pythagorean theorem is now: $V = \sqrt{V_x^2 + V_y^2 + V_z^2}$
- See the book for formulas for the components in terms of the two angles (we won't need them).

$$\boxed{V = V_x \, \hat{\imath} + V_y \, \hat{\jmath} + V_z \, \hat{k}}$$

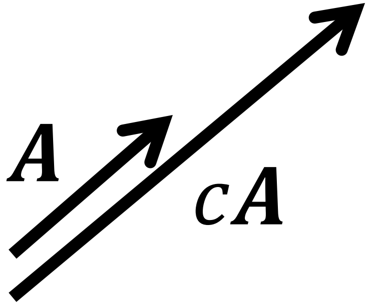
Operations on Vectors

- Given: Two vectors (***A*** and ***B***) and a positive number (c). What can we do with them?

1. Multiplication of a vector by a positive number: $\boldsymbol{A} \rightarrow c\boldsymbol{A}$.

This rescales length and components of $\boldsymbol{A}$ by a factor of c with no effect on direction:

$$
\begin{gathered}
A \rightarrow cA \\
A_x \rightarrow cAx \\
A_y \rightarrow cAy \\
\theta \rightarrow \theta
\end{gathered}
$$



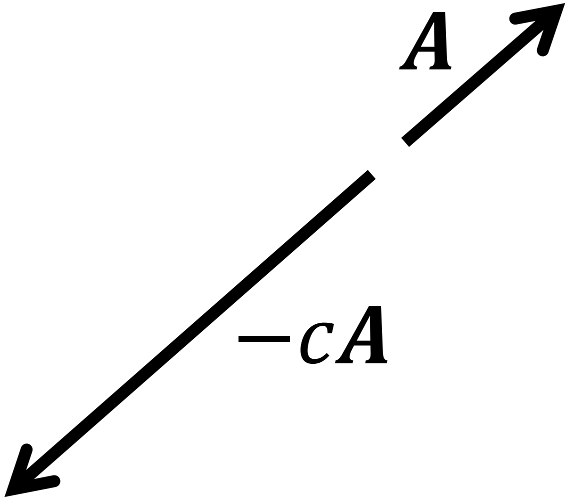
Operations on Vectors

2. Multiplication of a vector by a negative number:

$$\boldsymbol{A} \rightarrow -c\boldsymbol{A}\,.$$

This rescales length of $\boldsymbol{A}$ by a factor of c and components by $-c$. Direction is flipped by $180°$.

$$A \rightarrow cA$$
$$A_x \rightarrow -cA_x$$
$$A_y \rightarrow -cAy$$
$$\theta \rightarrow \theta + 180°$$

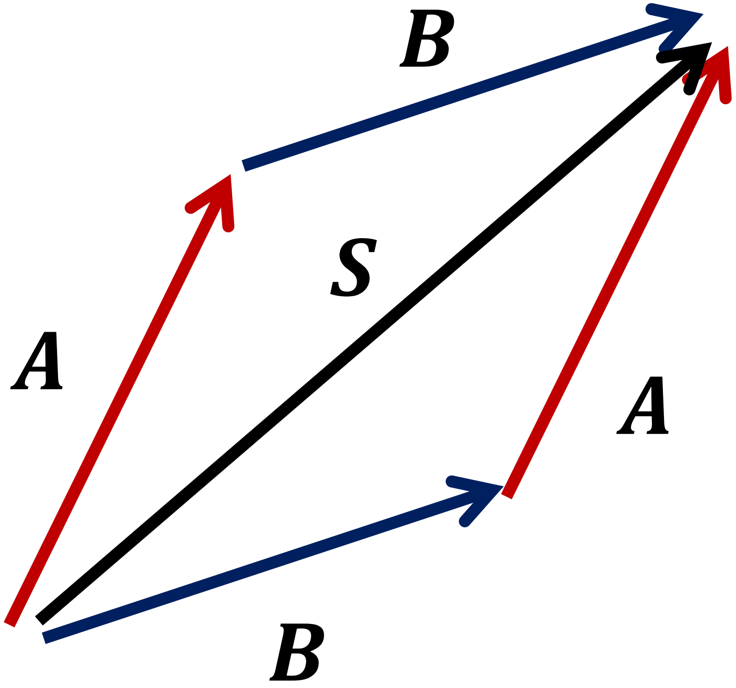


Operations on Vectors

3. Addition of two vectors: $\boldsymbol{S} = \boldsymbol{A} + \boldsymbol{B}$

Move one vector so its tail is at the tip of the other (parallelogram rule).

$$\boxed{\begin{aligned} S_x &= A_x + Bx \\ S_y &= A_y + B_y \end{aligned}}$$



Note: you can only add components,
you **CANNOT** add angles or magnitudes:

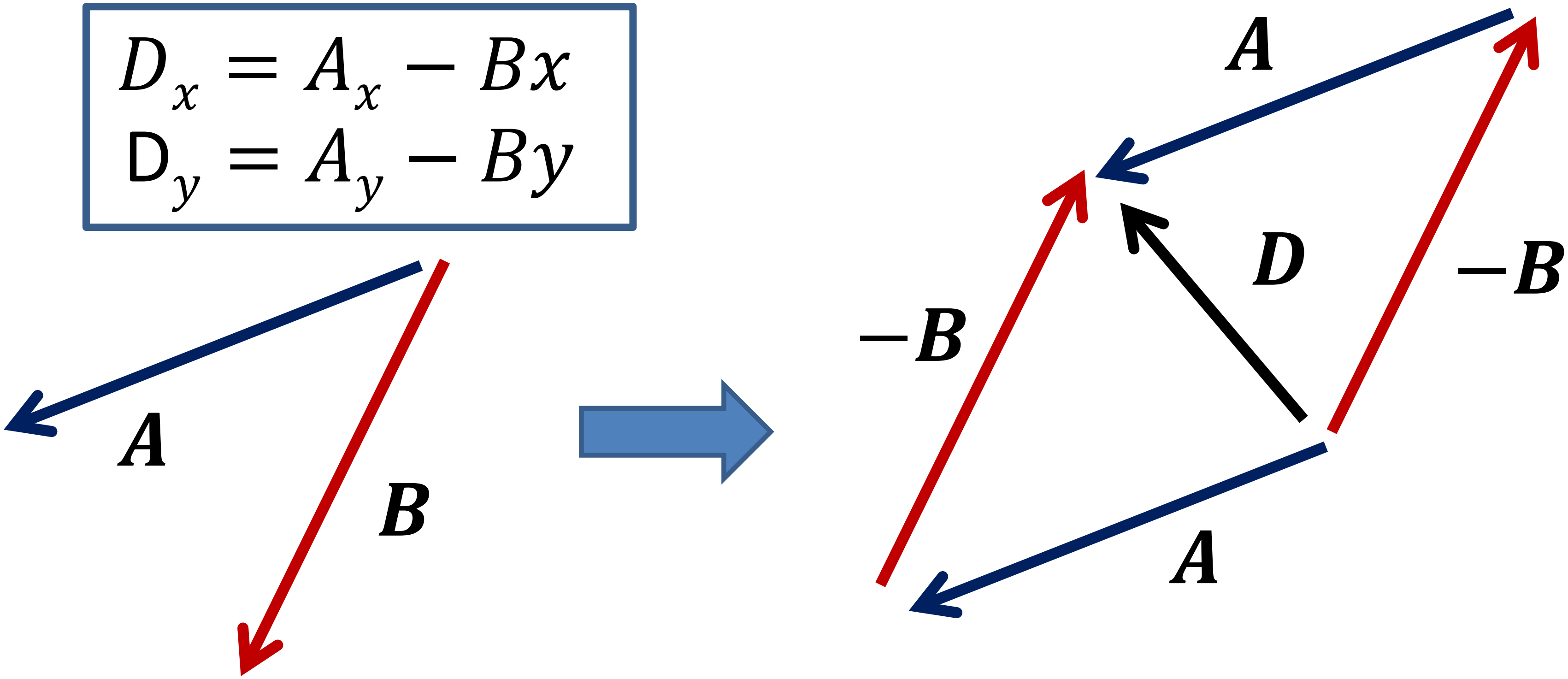
$$\theta_S \neq \theta_A + \theta_B$$
$$S \neq A + B$$

Operations on Vectors

3. Subtraction of two vectors: $\boldsymbol{D} = \boldsymbol{A} - \boldsymbol{B}$

Add the vector $-\boldsymbol{B}$ to the vector $\boldsymbol{A}$.

$$D_x = A_x - Bx$$
$$\mathrm{D}_y = A_y - By$$



Operations on Vectors

4. Multiply vectors times vectors.

There are two types of product:

- The cross product or vector product:

$$\boldsymbol{A} \times \boldsymbol{B} = a\ new\ vector$$

- The dot product or scalar product:

$$\boldsymbol{A} \cdot \boldsymbol{B} = a\ scalar$$

We’ll come back to these later.

Operations on Vectors

What you **CANNOT** do:

1. You **cannot** add or subtract scalars with vectors: $c \pm \boldsymbol{A}$
2. You **cannot** divide one vector into another:

$$\boldsymbol{A}/\boldsymbol{B}$$

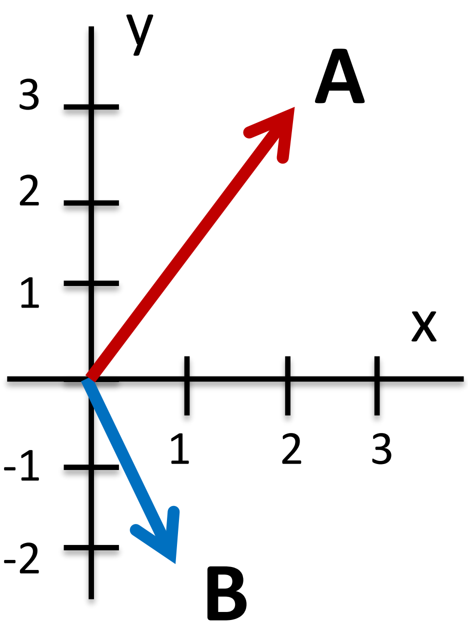
Once again: You also **cannot** add the magnitudes of vectors. To add vectors, you must break into components and **add the components**.

Example 1

- Two vectors A and B are given, as shown.

(a) Find the magnitude and direction of each vector.

$$A_x = 2,\ A_y = 3,\ B_x = 1,\ B_y = -2$$



For **A**:

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{2^2 + 3^2} = \boxed{\sqrt{13}} \approx 3.61$$

$$\theta_A = \tan^{-1}\frac{A_y}{A_x} = \tan^{-1}\frac{3}{2} = \boxed{56.31°}$$

For **B**:

$$B = \sqrt{B_x^2 + B_y^2} = \sqrt{1^2 + (-2)^2} = \boxed{\sqrt{5}} \approx 2.236$$

$$\theta_B = \tan^{-1}\frac{B_y}{B_x} = \tan^{-1}\left(\frac{-2}{1}\right) = \boxed{-63.4°}$$

Continued

Example 1 (cont.)

(b) Sketch $\boldsymbol{C} = -3\boldsymbol{B}$ and find its components.

Components:

$$C_x = -3B_x = -3(1) = \boxed{-3}$$

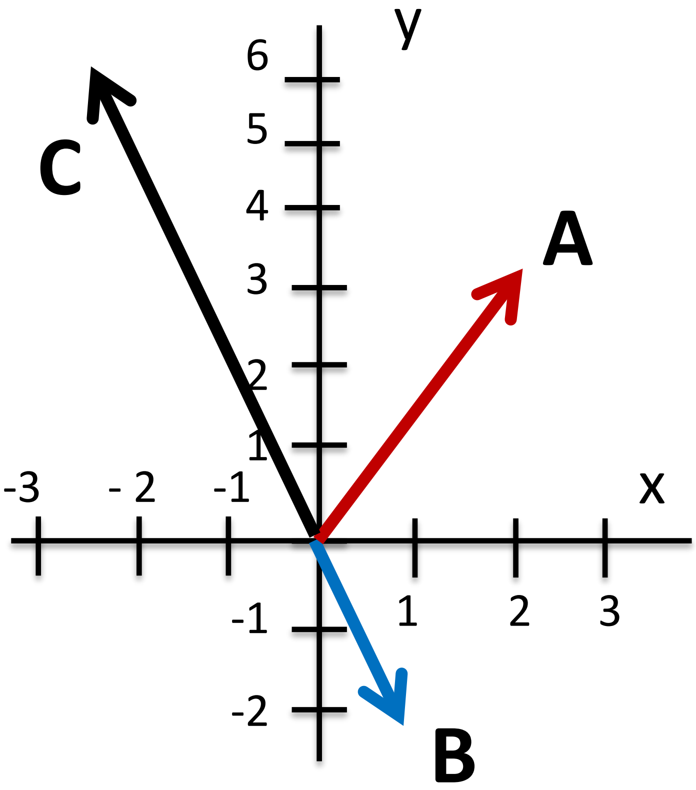
$$C_y = -3B_y = -3(-2) = \boxed{+6}$$

Also:

$$C = 3|B| = 3\sqrt{5}$$

$$\theta_C = \theta_B + 180°$$

$$= -63.4° + 180° = 117.6°$$

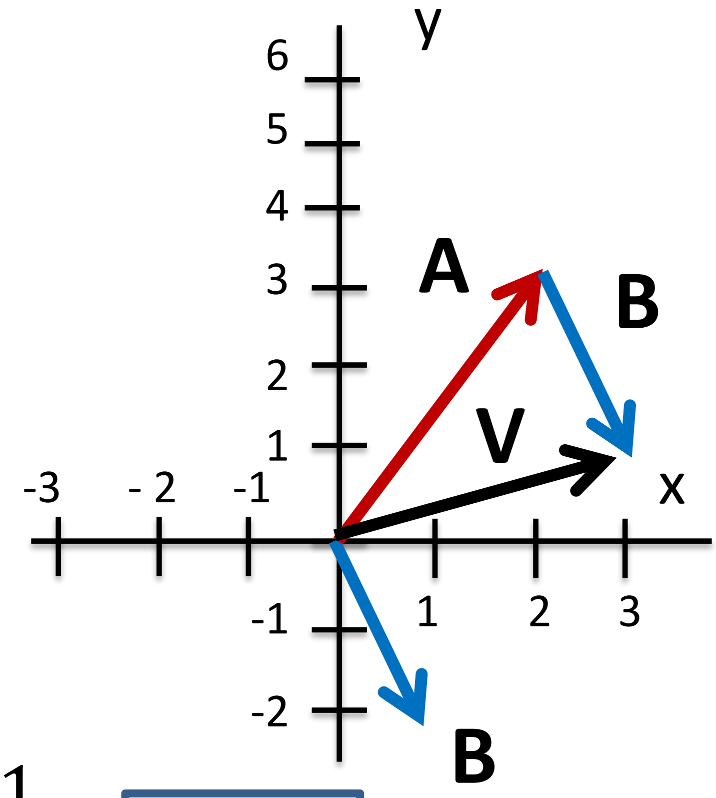


y
6
5
4
3
2
1
C
A
x
-3
- 2
-1
1
2
3
-1
-2
B

Continued

Example 1 (cont.)

(b) Find magnitude and direction of $\boldsymbol{V} = \boldsymbol{A} + \boldsymbol{B}$.



Components:

$$V_x = A_x + B_x = 2 + 1 = 3$$

$$V_y = A_y + B_y = 3 + (-2) = 1$$

Magnitude & direction:

$$V = \sqrt{V_x^2 + V_y^2} = \sqrt{3^2 + 1^2}$$

$$= \boxed{\sqrt{10}} \approx 3.16$$

$$\theta_V = \tan^{-1}\frac{V_y}{V_x} = \tan^{-1}\frac{V_y}{V_x} = \tan^{-1}\frac{1}{3} = \boxed{18.43°}$$

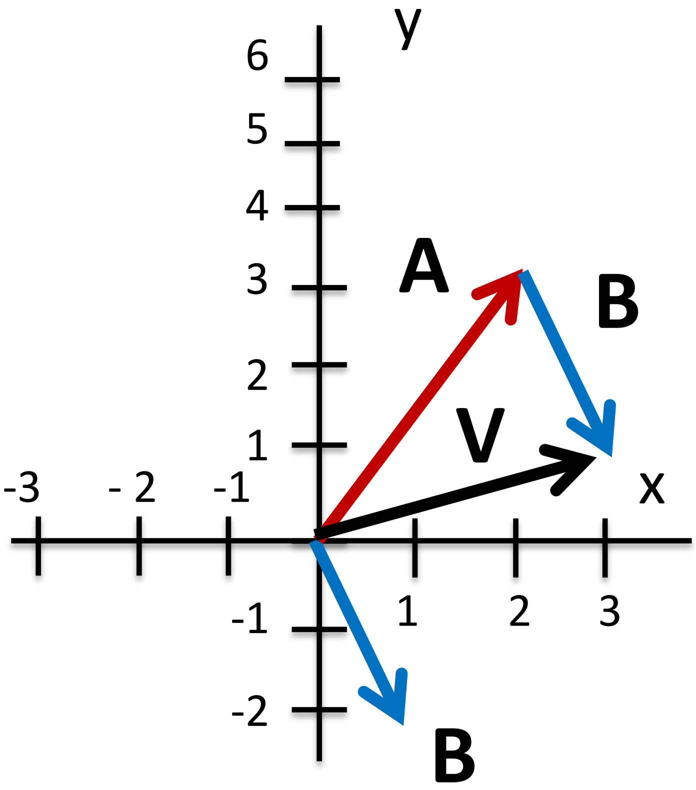
Continued

Example 1 (cont.)

Again - Notice that angles and lengths cannot be added:

$$\underset{\underset{18.43^\circ}{\nearrow}}{\theta_V} \neq \underset{\underset{-63.4^\circ}{\uparrow}}{\theta_A} + \underset{\underset{56.31^\circ}{\uparrow}}{\theta_B}$$

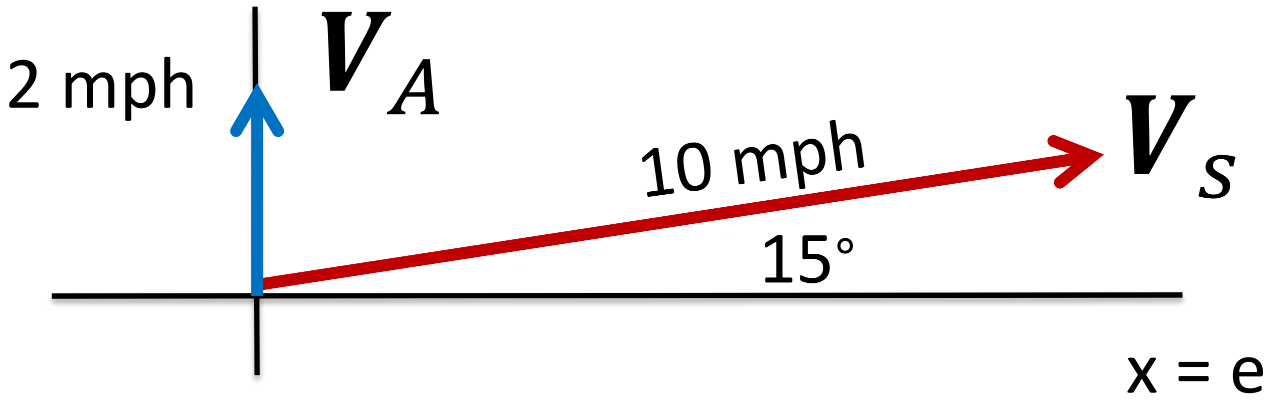
$$\underset{\underset{3.16}{\nearrow}}{V} \neq \underset{\underset{3.61}{\uparrow}}{A} + \underset{\underset{2.236}{\uparrow}}{B}$$



Example 2

- A sailboat is traveling at 10 mph, 15° N of E. A gust of wind gives it an additional velocity of 2 mph due north.
- What is the total final velocity?

Total: $\boldsymbol{V} = \boldsymbol{V}_S + \boldsymbol{V}_A$


y = north
2 mph
$\boldsymbol{V}_A$
10 mph
$\boldsymbol{V}_S$
15°
x = east

Example 2 (continued)

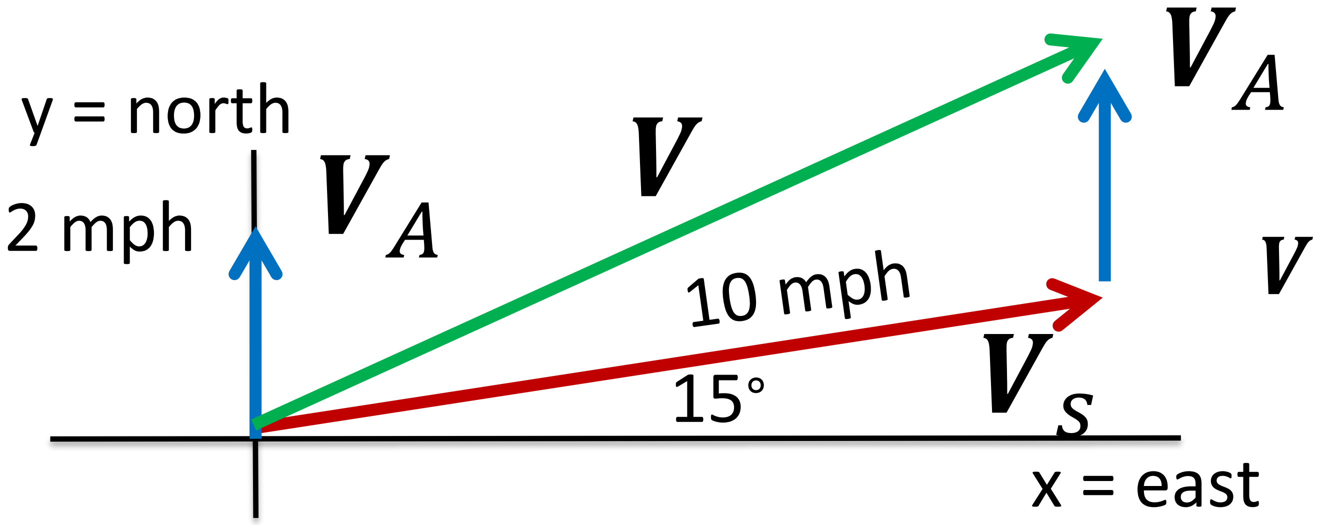
- To add the vectors, we first need components:

$$V_{Sx} = V_S \cos 15° = 10 \cos 15° = 9.66\, mph$$

$$V_{Sy} = V_S \sin 15° = 10 \sin 15° = 2.59\, mph$$

$$V_{Ax} = V_A \cos 90° = 2 \cos 90° = 2\, mph$$

$$V_{Ay} = V_A \sin 90° = 2 \sin 90° = 0$$



$\boldsymbol{V_A}$
y = north
$\boldsymbol{V}$
2 mph
$\boldsymbol{V_A}$
10 mph
15°
$\boldsymbol{V_S}$
x = east

$$\boldsymbol{V} = \boldsymbol{V_S} + \boldsymbol{V_A}$$

Example 2 (continued)

- Total: $V_x = V_{Sx} + V_{Ax} = 9.66 + 0 = 9.66\,mph$

  $V_y = V_{Sy} + V_{Ay} = 2.59 + 2 = 4.59\,mph$

- Magnitude and direction:

$$V = \sqrt{V_x^2 + V_y^2} = \sqrt{9.66^2 + 4.59^2} = \boxed{10.7\,mph}$$

$$\theta_V = \tan^{-1}\frac{V_y}{V_x} = \tan^{-1}\frac{4.59}{9.66} = \boxed{25.41°}$$

Products of Vectors

- There are two completely different types of products that can be defined for vectors.
- Suppose we are given two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. Then we can form:
  - The **dot product** or **scalar product**: $\boldsymbol{A} \cdot \boldsymbol{B}$
  - The **cross product** or **vector product**: $\boldsymbol{A} \times \boldsymbol{B}$
- **Examples** of each in physics:
  - **Dot products:** work, electric force on dipole
  - **Cross product:** torque, magnetic forces

Dot Products

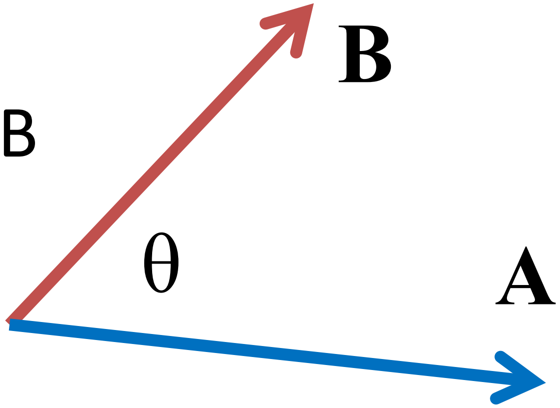
- Given two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, we can construct a product called the **dot product** or **scalar product**: $\boldsymbol{A} \cdot \boldsymbol{B}$
- The result of this product is a **scalar**: it has no components or direction.
- The order of the multiplication doesn't matter: $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$
- The dot product can be positive or negative (or zero)

Dot Products

- There are two equivalent formulas for the dot product:

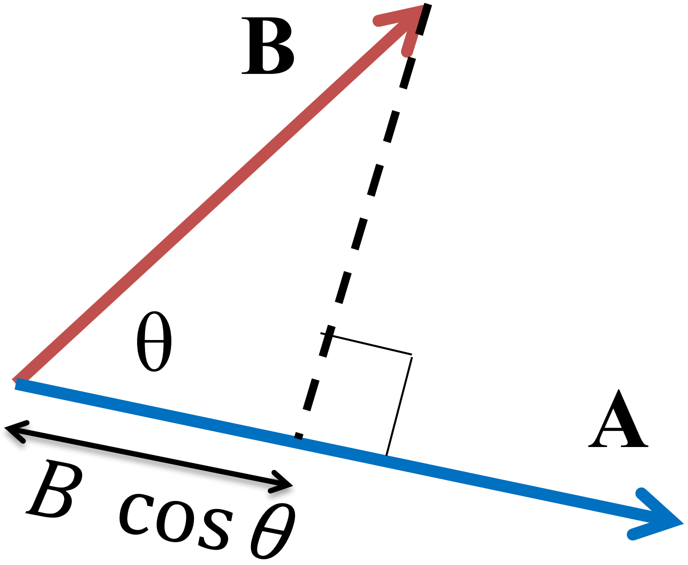
$$\boxed{\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{B} &= A_xB_x + A_yB_y \\ \boldsymbol{A}\cdot\boldsymbol{B} &= AB\ \cos\theta \end{aligned}}$$

- θ is the angle from between **A** and **B**, **not** the angle from the x-axis.
- In the second equation, the factor AB on the right is the product of the **magnitudes**.
- In three dimensions, the first equation becomes: $\boldsymbol{A}\cdot\boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z$



Dot Products

- The formula $\boldsymbol{A} \cdot \boldsymbol{B} = AB\ \cos\theta$ has a simple geometric meaning.
- $\boldsymbol{B} \cos \boldsymbol{\theta}$ is the **projection** or **component** of ***B*** in the direction of ***A***
- The dot product multiplies the parts of A and B that are parallel to each other.
- The perpendicular parts don't contribute to the product.



Example 1

- Suppose two vectors of lengths $A = 3$ and $B = 4$ are separated from each other by 25°.
- Find $\boldsymbol{A} \cdot \boldsymbol{B}$.

We are given lengths and the angle between the vectors, so use second formula.

$$\boldsymbol{A} \cdot \boldsymbol{B} = \mathrm{AB} \cos \theta = (3)(4) \cos 25° = \boxed{10.9}$$

Example 2

- Given two vectors: $\boldsymbol{A} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}, \quad \boldsymbol{B} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

(a) Find $\boldsymbol{A} \cdot \boldsymbol{B}$.

$$\begin{aligned}\boldsymbol{A} \cdot \boldsymbol{B} &= A_x B_x + A_y B_y + A_z B_z \\ &= (1)(2) + (-1)(-3) + (1)(2) = \boxed{7}\end{aligned}$$

(b) Find the angle between **A** and **B**.

$$\mathrm{A} = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\mathrm{B} = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{2^2 + (-3)^2 + 2^2} = \sqrt{17}$$

$$\boldsymbol{A} \cdot \boldsymbol{B} = AB \cos\theta \quad \Longrightarrow \quad \theta = \cos^{-1}\frac{\boldsymbol{A} \cdot \boldsymbol{B}}{AB} = \cos^{-1}\frac{\boldsymbol{7}}{\sqrt{3}\sqrt{17}}$$

$$= \boxed{11.47°}$$

Cross product

- There is a second type of product for vectors: the cross product or vector product, $\boldsymbol{A} \times \boldsymbol{B}$.
- Unlike the dot product the cross product results in a new vector.
- It is also **non-commutative** – order matters:

$$\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$$

Cross product

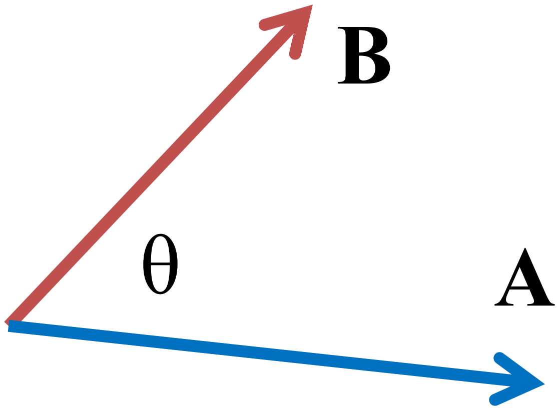
- Definition of cross product:

$$\boldsymbol{A}\times\boldsymbol{B} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}$$

- Magnitude of cross product:

$$|\boldsymbol{A}\times\boldsymbol{B}| = AB\sin\theta$$

(where θ is angle between **A** and **B**)

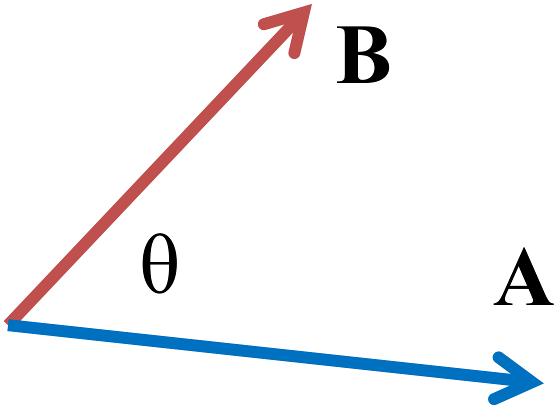


Cross product

- Magnitude of cross product:

$$\boxed{|\boldsymbol{A} \times \boldsymbol{B}| = AB\cos\theta}$$

(where θ is angle between **A** and **B**)



- Compare to dot product: $\boldsymbol{A} \cdot \boldsymbol{B} = AB\sin\theta$
- The dot product multiplies the parts of A and B that are parallel to each other. The cross product multiplies the perpendicular parts.
- Note: $\boldsymbol{A} \times \boldsymbol{B} = 0$ if A and B are parallel ($\theta = 0$)

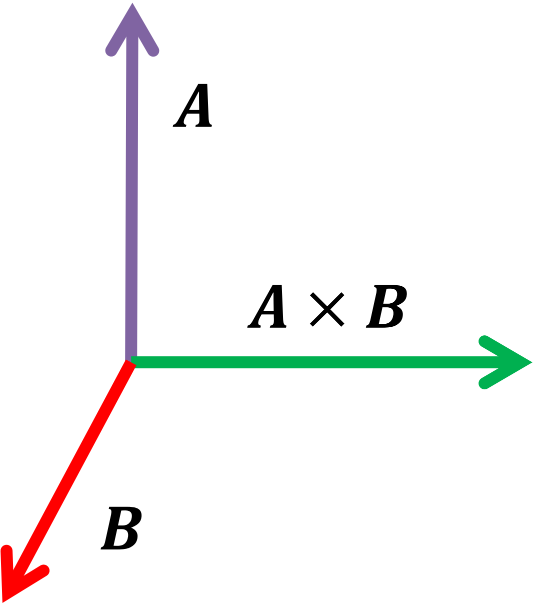
Cross product: right-hand rule

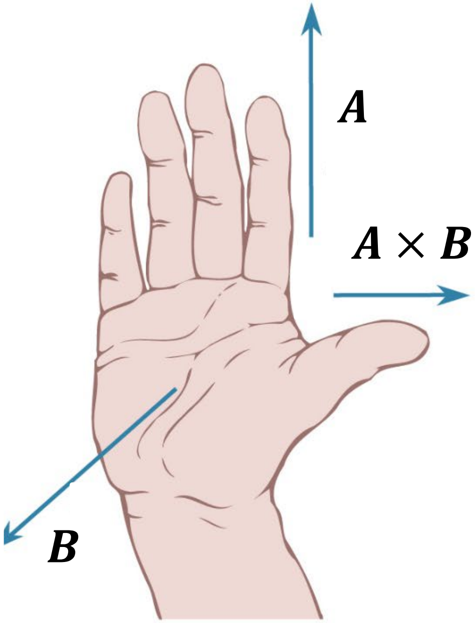
- The direction of $\boldsymbol{A} \times \boldsymbol{B}$ is always perpendicular to both $\boldsymbol{A}$ and $\boldsymbol{B}$, and is given by the **right-hand rule.**
- Using **right** hand:

➢ First vector ($\boldsymbol{A}$) along fingers

➢ Palm in direction of 2^{nd} vector ($\boldsymbol{B}$)

➢ $\boldsymbol{A} \times \boldsymbol{B}$ is along thumb (⊥ to both $\boldsymbol{A}$ and $\boldsymbol{B}$)

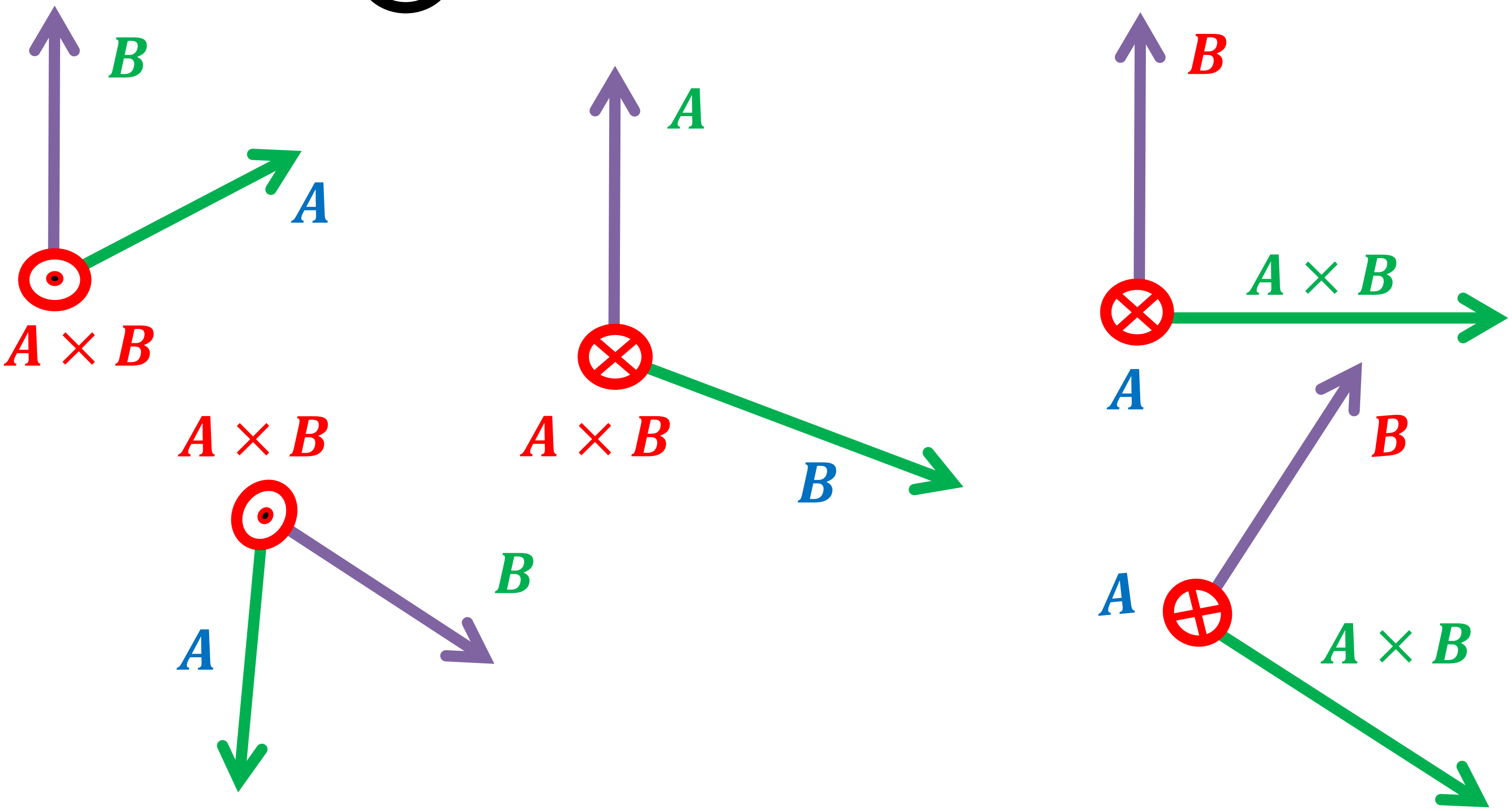

$\boldsymbol{A}$
$\boldsymbol{A} \times \boldsymbol{B}$
$\boldsymbol{B}$


$\boldsymbol{A}$
$\boldsymbol{A} \times \boldsymbol{B}$
$\boldsymbol{B}$

Cross product: right-hand rule

- Notation: ⊙ = pointing into page (or screen, etc)

  ⊗ = pointing out of page (or screen, etc)



B
A
$A \times B$
A
$A \times B$
B
B
$A \times B$
A
A
B
A
$A \times B$
$A \times B$
B
A

Example

- Suppose $\boldsymbol{A} = 3\boldsymbol{i} - 4\boldsymbol{j}$ and $\boldsymbol{B} = -2\boldsymbol{i} + 3\boldsymbol{k}$.

$$\boldsymbol{A} \times \boldsymbol{B} = \left(A_y B_z - A_z B_y\right)\mathbf{i} + (A_z B_x - A_x B_z)\mathbf{j} + \left(A_x B_y - A_y B_x\right)\mathbf{k}$$

$$= [(-4)(3) - (-4)(0)]\,\mathbf{i} + [(0)(-2) - (3)(3)]\,\mathbf{j} + [(3)(0) - (-4)(-2)]\,\mathbf{k}$$

$$= \boxed{-12\,i - 9\,j - 8\,k}$$

Magnitude: $$|\boldsymbol{A} \times \boldsymbol{B}| = \sqrt{(\boldsymbol{A} \times \boldsymbol{B})_x^2 + (\boldsymbol{A} \times \boldsymbol{B})_y^2 + (\boldsymbol{A} \times \boldsymbol{B})_z^2}$$

$$= \sqrt{(-12)^2 + (-9)^2 + (-8)^2} = \boxed{17}$$

Continued

Example (continued)

- What is the angle between $\boldsymbol{A} = 3\boldsymbol{i} - 4\boldsymbol{j}$ and $\boldsymbol{B} = -2\boldsymbol{i} + 3\boldsymbol{k}$?

We know that: $|\boldsymbol{A} \times \boldsymbol{B}| = AB \sin\theta$

So:

$$\theta = \sin^{-1}\left(\frac{|\boldsymbol{A} \times \boldsymbol{B}|}{A\,B}\right) = \sin^{-1}\left(\frac{\mathbf{17}}{5\sqrt{13}}\right) = \boxed{70.56°}$$

$$|A| = A = \sqrt{A_x^2 + A_y^2} = \sqrt{3^2 + 4^2} = 5$$

$$|B| = B = \sqrt{B_x^2 + B_y^2} = \sqrt{2^2 + 3^2} = \sqrt{17}$$

Two dimensional motion

- In two dimensions, position and velocity are vectors and each have two components.

$$Position: \quad \boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$$

$$Velocity: \quad \boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j}$$

- Similarly, acceleration due to gravity is a vector: $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j},$

  where:

$$a_x = 0,$$

$$a_y = -g = -9.8\ \frac{m}{s^2}$$

Projectile Motion

- Consider an object shot into the air, with initial speed v_0, at angle θ from the horizontal.
- What are the position & velocity as a function of time?
- Only acceleration is that due to gravity.
- Position and velocity are both vectors:

Initial	**Final**
$\boldsymbol{r_0} = x_0\boldsymbol{i} + y_0\boldsymbol{j}$	$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$
$\boldsymbol{v_0} = v_{x0}\boldsymbol{i} + v_{y0}\boldsymbol{j}$	$\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j},$

where: $v_0 = \sqrt{v_{x0}^2 + v_{y0}^2}$

Projectile Motion

- Both components of the acceleration are constant.
- To describe projectile motion we simply use two copies of the constant acceleration equations:
  - One for the horizontal direction (with $a_x = 0$)
  - One for the vertical direction (with $a_y = -g$)

Projectile Motion

- General equations:

1. $x = x_0 + v_0 t + \frac{1}{2} a t^2$
2. $v = v_0 + at$
3. $v^2 = v_0^2 + 2a(x - x_0)$

$a_x = 0$

$a_y = -g$

Horizontal (x)

1. $x = x_0 + v_{x0} t$
2. $v_x = v_{x0}$
3. $v_x^2 = v_{x0}^2$

Vertical (y)

1. y$= y_0 + v_{y0} t - \frac{1}{2} g t^2$
2. $v_y = v_{y0} - gt$
3. $v_y^2 = v_{y0}^2 - 2g(y - y_0)$

Projectile Motion

- General equations:

1. $x = x_0 + v_0 t + \frac{1}{2} a t^2$
2. $v = v_0 + at$
3. $v^2 = v_0^2 + 2a(x - x_0)$

$a_x = 0$

$a_y = -g$

Horizontal (x)

1. $x = x_0 + v_{x0} t$
2. $v_x = v_{x0}$
3. $\xcancel{v_x^2 = v_{x0}^2}$

Vertical (y)

1. $y = y_0 + v_{y0} t - \frac{1}{2} g t^2$
2. $v_y = v_{y0} - gt$
3. $v_y^2 = v_{y0}^2 - 2g(y - y_0)$

Notice this equation is redundant.

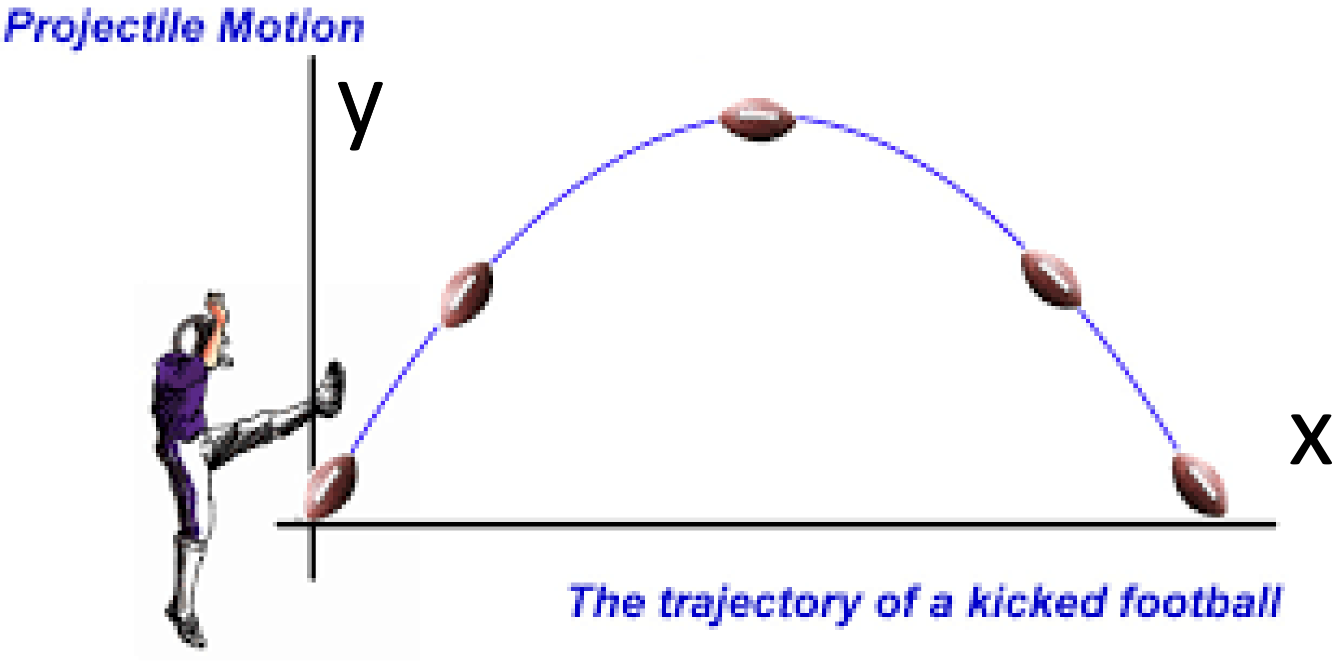
Projectile Motion

- So our main equations are:

Horizontal (x)	Vertical (y)
1. $x = x_0 + v_{x0}t$	1. $y = y_0 + v_{y0}t - \frac{1}{2}gt^2$
2. $v_x = v_{x0}$	2. $v_y = v_{y0} - gt$
	3. $v_y^2 = v_{y0}^2 - 2g(y - y_0)$

The motion of a projectile is parabolic →

Projectile: Trajectory is a Parabola

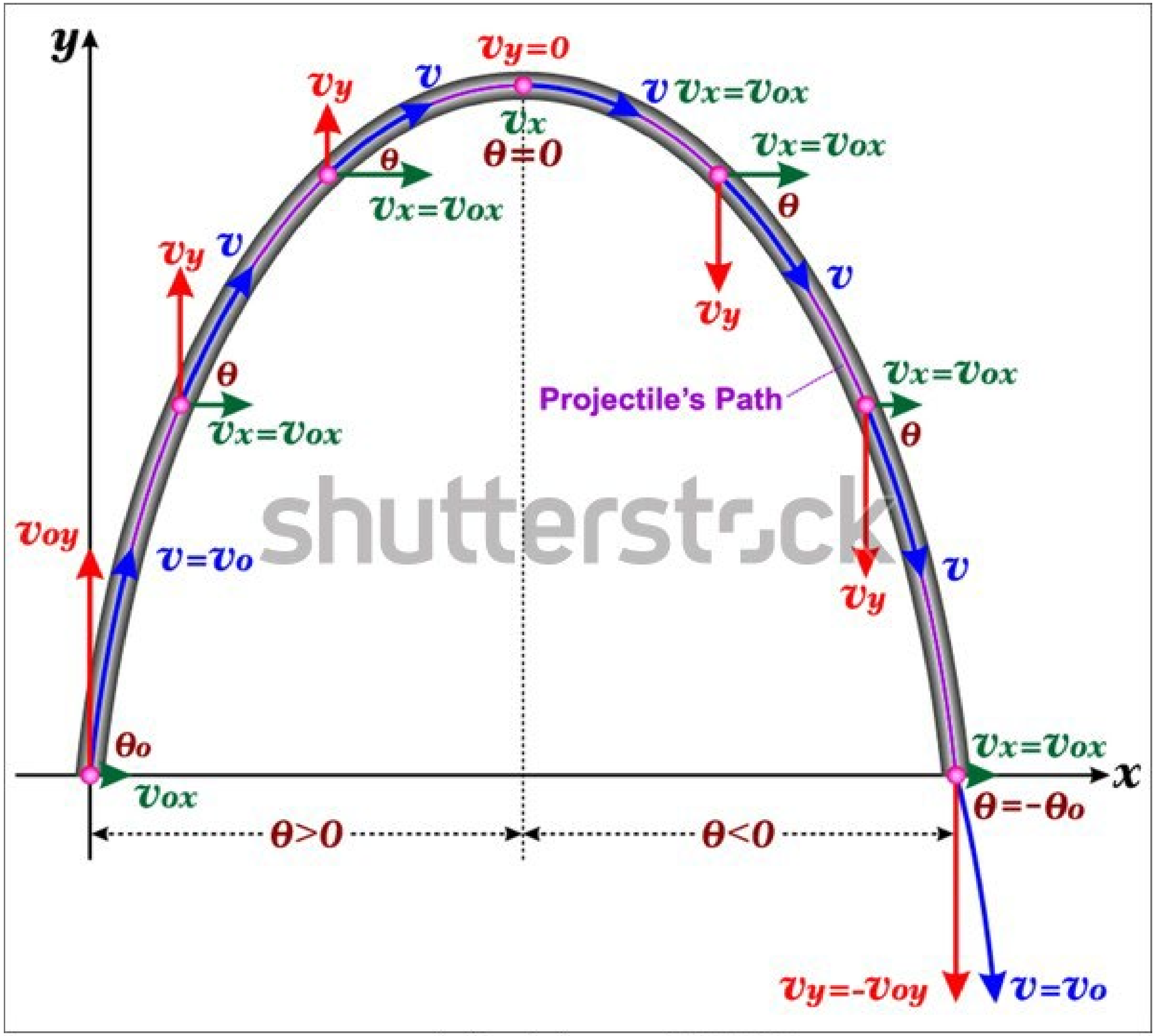

Projectile Motion
y
x
The trajectory of a kicked football

Plugging 2nd x equation into 2nd y equation to eliminate time, we find equation of parabola:

$$y = (\tan\theta)x - \left(\frac{g}{2v_0^2 cos^2\theta}\right)x^2$$

(General equation for parabola: $y = a\,x^2 + b\,x + c$)

Anatomy of Parabolic Motion



y
vy=0
v
vy
v Vx=Vox
Vx
θ=0
θ
Vx=Vox
Vx=Vox
θ
vy
v
v
vy
θ
Vx=Vox
Projectile's Path
Vx=Vox
θ
Voy
v=Vo
v
vy
θo
Vox
Vx=Vox
x
θ=-θo
θ>0
θ<0
vy=-Voy
v=Vo

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- v starts positive, decreases over time, eventually becoming negative
- a is **always** negative (it is constant)
- At maximum height, $v_y = 0$.
- v_x is **not** zero at maximum height (remember it is constant), and **speed** is not zero.

Example 1

- A stuntperson drives motorcycle off cliff with initially horizontal velocity of 5 m/s. Find position and velocity after .25 s.
- Given: $v_{0x} = 5\frac{m}{s}, v_{0y} = 0,\ t = .25s$
- Define: $x_0 = 0, y_0 = 0$
- Want: $x,\ y, v_x, v_y$ at $t = .25s$

Example 1 (continued)

- Find position first:

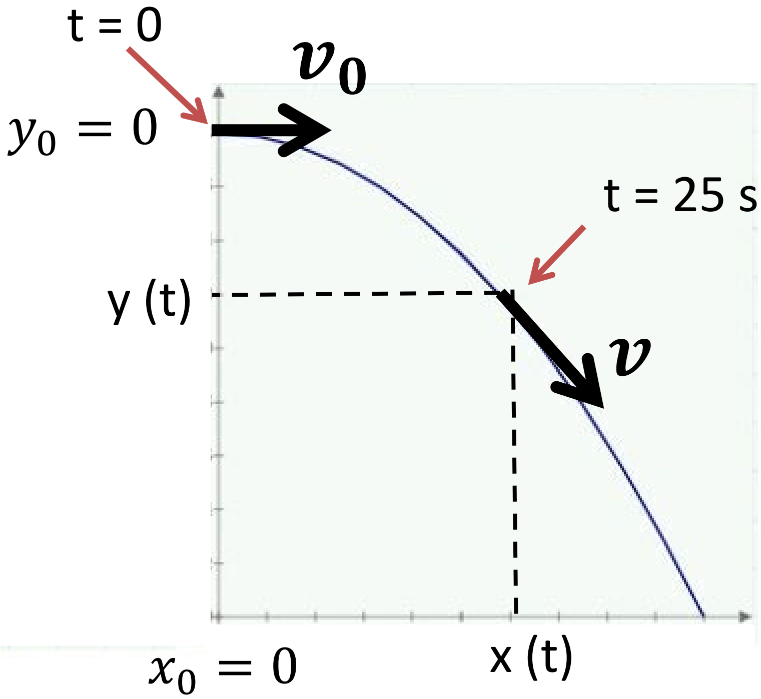
$$x = x_0 + v_{ox}t = 0 + \left(5\ \frac{m}{s}\right)(.25\ s)$$

$$= \boxed{1.25\ m}$$

$$y = y_0 + v_{oy}t - \frac{1}{2}gt^2$$

$$= 0 + 0 - \frac{1}{2}(9.8\frac{m}{s^2})(.25\ s)^2$$

$$= \boxed{-.306\ m}$$



t = 0
v_0
$y_0 = 0$
t = 25 s
y (t)
v
$x_0 = 0$
x (t)

- Now find velocity

Example 1 (continued)

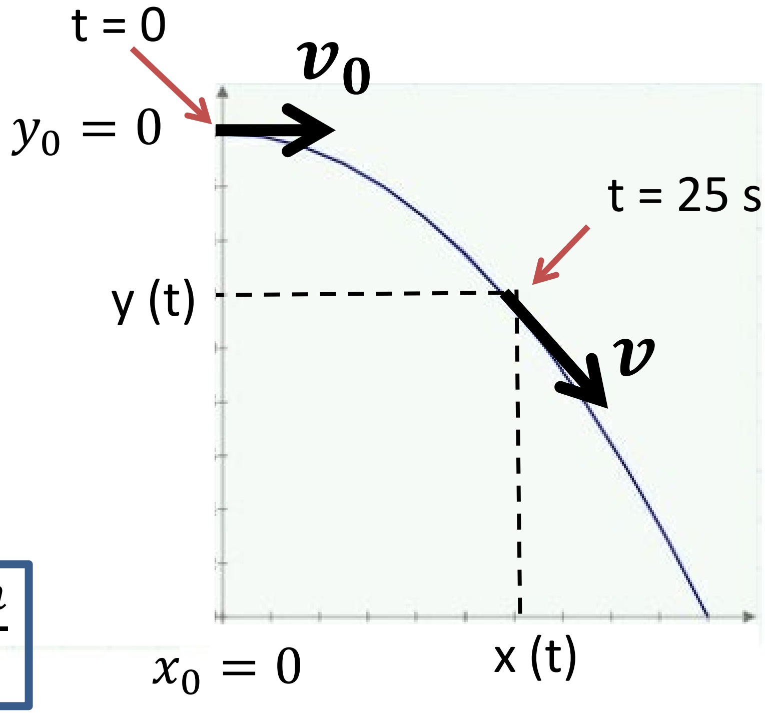
- Find velocity:

$$v_x = v_{0x} = \boxed{5\,\frac{m}{s}}$$

(since $a_x = 0$)

$$v_y = v_{0y} - gt$$

$$= 0 - \left(9.8\,\frac{m}{s^2}\right)(.25\ s) = \boxed{-2.45\,\frac{m}{s}}$$


t = 0
$\boldsymbol{v_0}$
$y_0 = 0$
t = 25 s
y (t)
$\boldsymbol{v}$
$x_0 = 0$
x (t)

- Speed: $v = \sqrt{v_x^2 + v_y^2} = \sqrt{(5)^2 + (2.45)^2} = 5.57\,\frac{m}{s}$
- Direction: $\theta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\left(\frac{-2.45}{5}\right) = -26.1°$

Example 2

- A ball is thrown into the air at $40°$ above the horizon with initial speed $v_0 = 3\ {}^{m}/_{s}$.

(a) What is the speed when the ball is .1 m above initial height $(y - y_0 = .1\, m)$.

First find components at t=0:

$$v_{0x} = v_0 \cos\theta = 3\ \cos 40° = 2.30\ {}^{m}/_{s}$$

$$v_{0y} = v_0 \sin\theta = 3\ \sin 40° = 1.93\ {}^{m}/_{s}$$

Find final components:

(v_y= 1.33 m/s)

↓

$$v_x^2 = v_{0x}^2 = (2.3\,{}^{m}/_{s})^2 = 5.29\,{}^{m^2}/_{s^2}$$

$$v_y^2 = v_{0y}^2 - 2g(y - y_0) = (1.93)^2 - 2(9.8)(.1) = 1.76\,{}^{m^2}/_{s^2}$$

Find final speed:

$$v = \sqrt{v_x^2 + v_x^2} = \sqrt{5.29 + 1.76} = \boxed{2.66\ \text{m/s}}$$

Example 2 (continued)

(b) Find the maximum height the ball reaches (as measured from initial height).

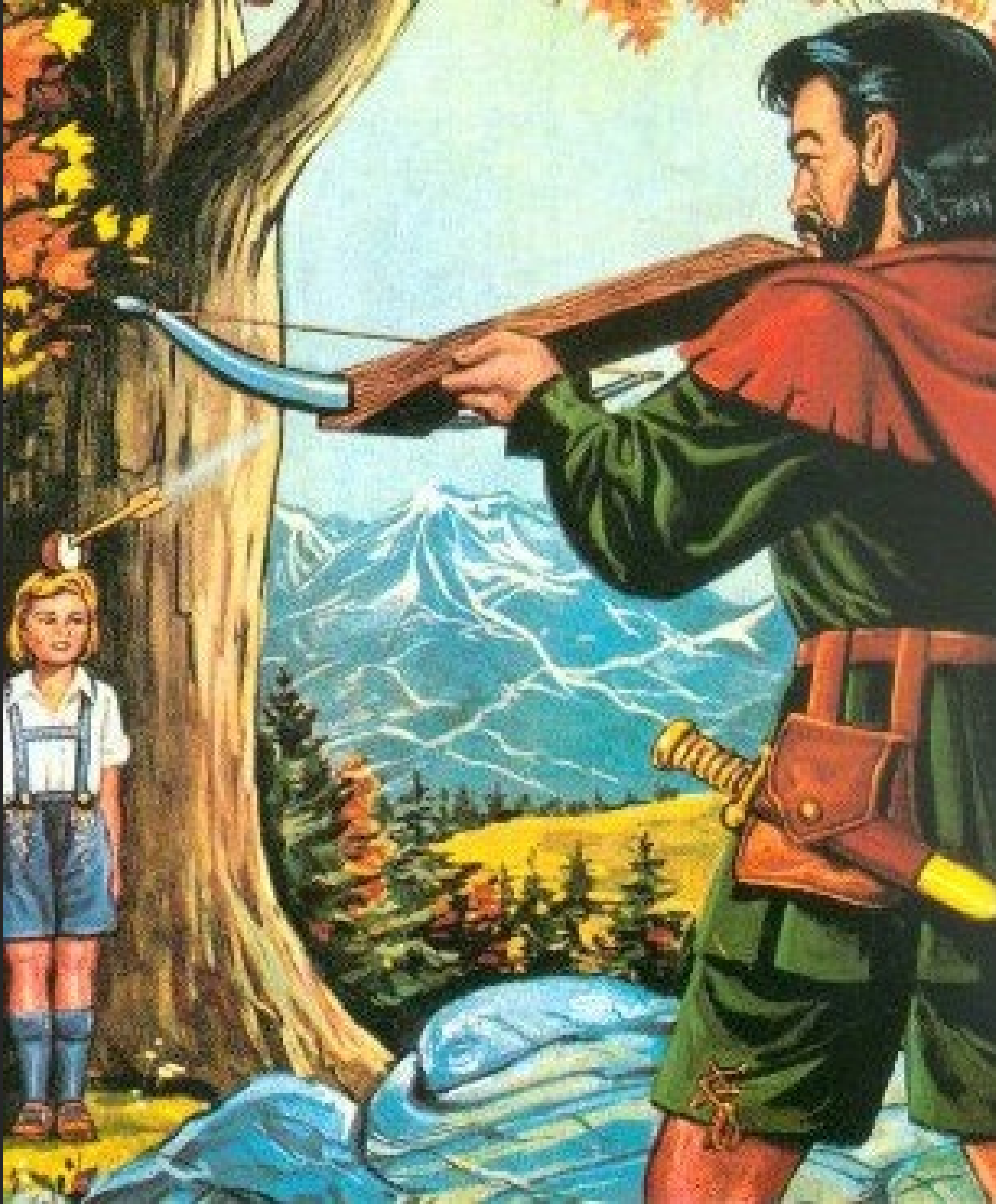
At maximum height, we know $v_y = 0$. So:

$$\cancel{v_y^2} = v_{0y}^2 - 2g(y - \cancel{y_0}) \quad \rightarrow \quad 0 = v_{0y}^2 - 2\mathrm{gy}$$

$$y = \frac{v_{0y}^2}{2g} = \frac{(1.93)^2}{(2)(9.8)} = \boxed{0.19\ m}$$

Note again that only v_y is zero at the maximum height, **not** v_x or the speed. Also, in part a, be careful not to confuse the components of the initial and final velocities.

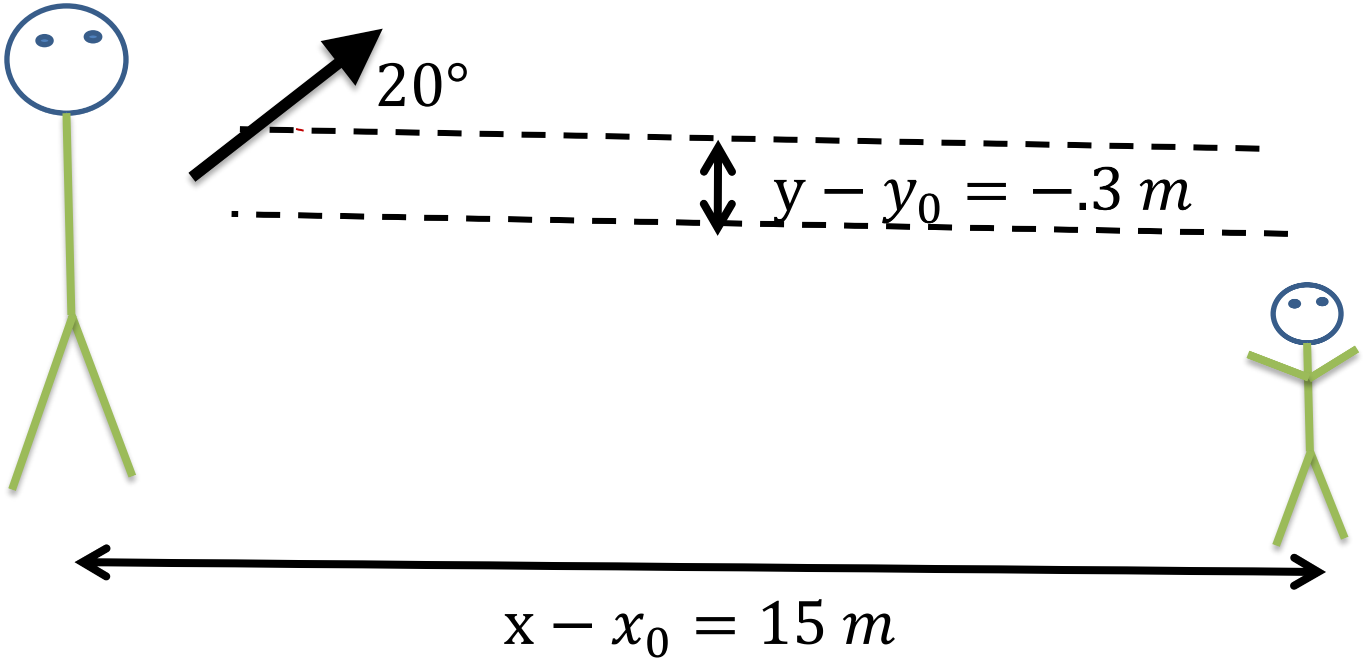
Example 3: William Tell and the apple



Disclaimer: no children (or innocent fruit) were harmed in this example.

Example 3

- William Tell aims arrow at an apple, 15 m away, on his son's head. The apple is .3 m below the initial height of the arrow. (Take $x_0 = y_0 = 0$.) Assume launch angle is 20° above the horizon.



Example 3 (continued)

(a) What must initial launch speed, v_0 , be?

Recall the parabola equation for the trajectory:

$$y = (\tan\theta)x - \left(\frac{g}{2v_0^2 cos^2\theta}\right)x^2$$

Solve for initial speed:

$$v_0 = \sqrt{\frac{gx^2}{2cos^2\theta(x\tan\theta - y)}} = \sqrt{\frac{(9.8)(15)^2}{2cos^2 20°((15)\tan 20° - (-.3))}}$$

$$v_0 = \boxed{14.72 \text{ m/s}}$$

Given:

$$y - y_0 = -.3\ m$$

$$x - x_0 = 15\ m$$

$$\theta = 20°$$

Example 3 (continued)

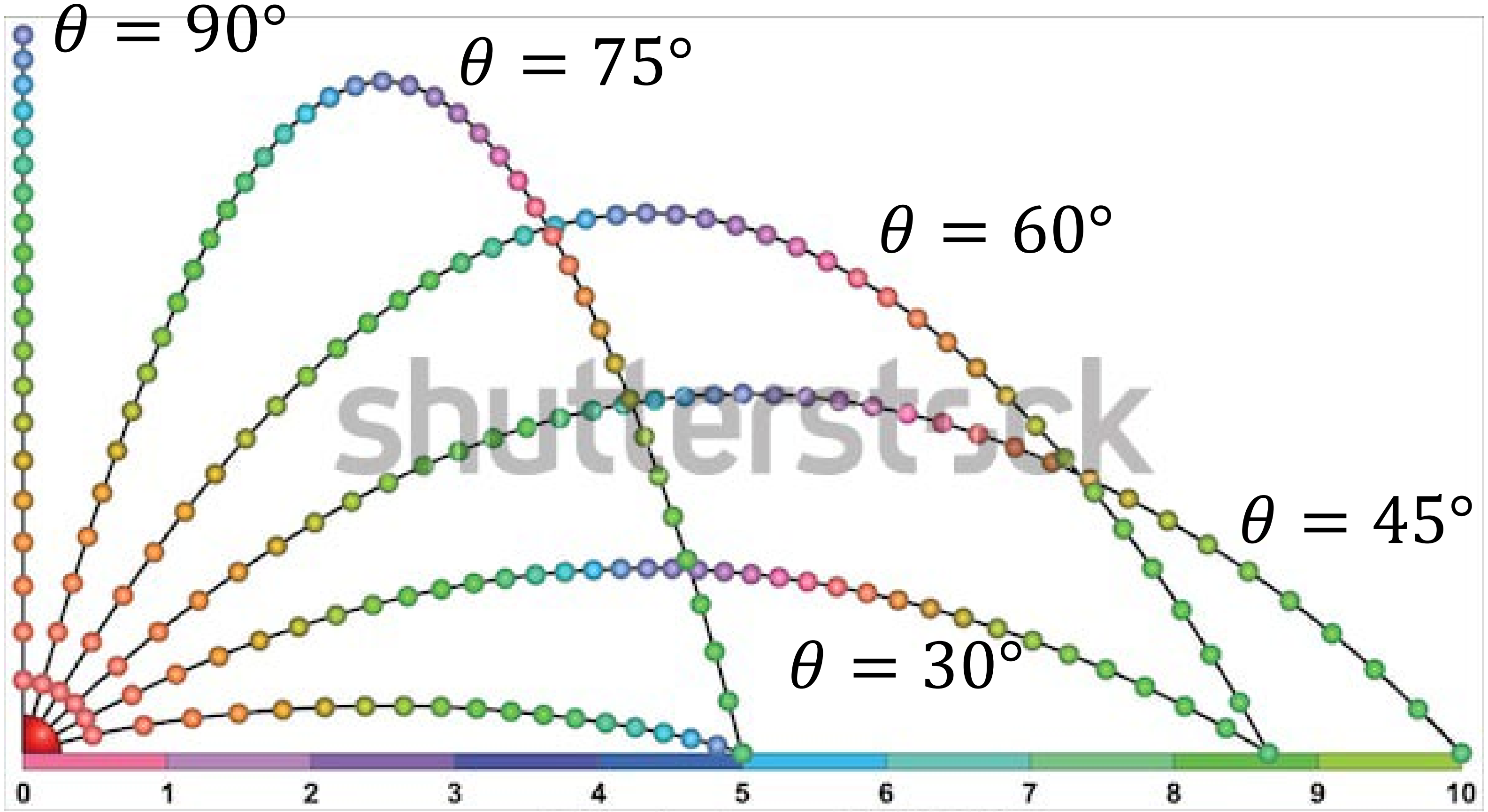
(b) How long is the arrow in the air?

$$x = v_{0x}t \quad \rightarrow \quad t = \frac{x}{v_{0x}} = \frac{x}{v_0 cos\theta} = \frac{15}{(14.72)\cos 20°}$$

$$t = \boxed{1.08\ s}$$

Height and distance

- For fixed launch speed, both height and distance traveled depend on launch angle



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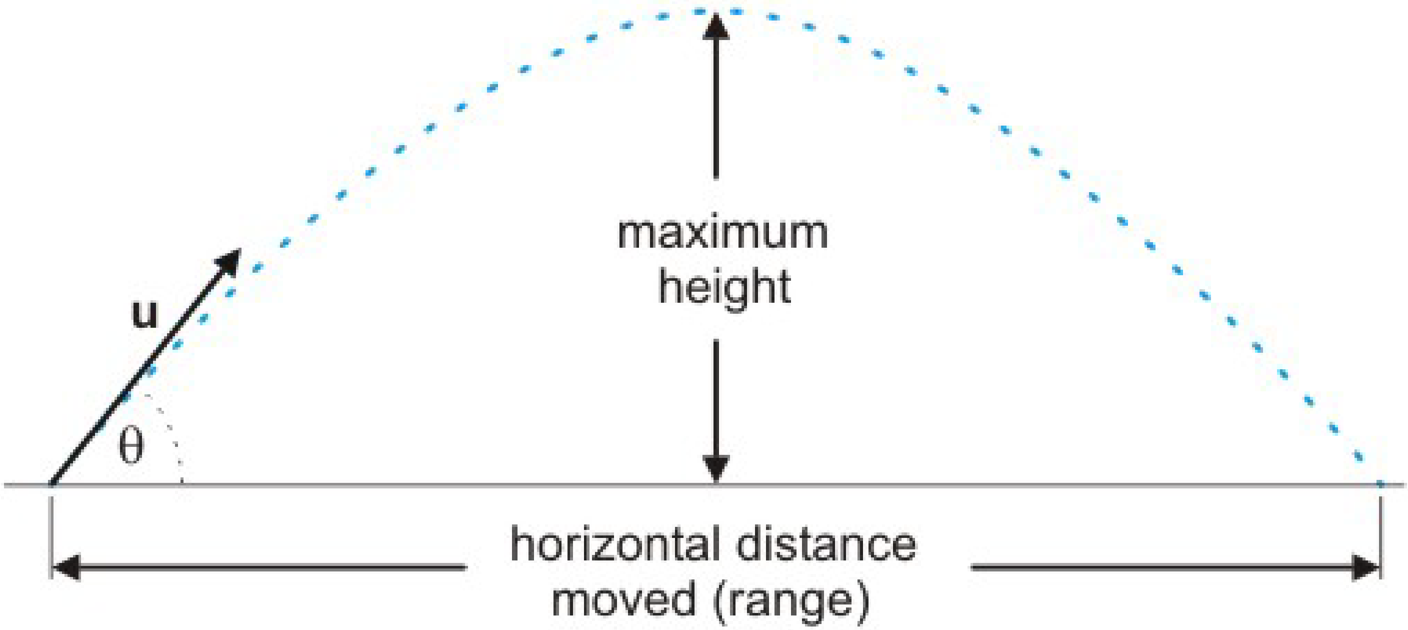
Range

- The **range R** is the distance a projectile travels horizontally before returning to its initial height.


u
θ
maximum height
horizontal distance moved (range)

Range

- To find R, set $y = y_0$ and solve for $R = x - x_0$.

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \xrightarrow{y = y_0} 0 = v_{0y}t - \frac{1}{2}gt^2$$

- Solve for t:

Launch point

Landing point

$$0 = \left(v_{0y} - \frac{1}{2}gt\right)t \quad \rightarrow \quad t = 0 \quad or \quad t = \frac{2v_{0y}}{g}$$

- Use t to find R:

$$R = x - x_0 = v_{0x}t = v_{0x}\left(\frac{2v_{0y}}{g}\right) = \left(\frac{2(v_0 \cos\theta)(v_0 \sin\theta)}{g}\right)$$

(Using $v_{0x} = v_0 \cos\theta$ and $v_{0y} = v_0 \sin\theta$)

- We can simplify this result

Range

- We have: $R = \left(\frac{2v_0^2 \cos\theta \; \sin\theta}{g}\right)$

- But recall the trig identity: $\sin 2\theta = 2\sin\theta\cos\theta$

- So: $\boxed{R = \left(\frac{v_0^2 \sin 2\theta}{g}\right)}$

- **Dire Warning:** **Do not use this formula unless initial and final heights are the same.**

Maximum Range

For given v_0, maximum range occurs at $\boldsymbol{\theta = 45^\circ}$

- **Proof 1:** The range is given by $R = \left(\frac{v_0^2 \sin 2\theta}{g}\right)$. But v_0 and g are fixed, so R is maximum when $\sin 2\theta$ is maximum. The maximum of the sine function occurs when $2\theta = 90^\circ$ $(\sin 90^\circ = 1)$, so: $\boxed{\theta = 45^\circ}$.
- **Proof 2:** Maximum occurs when $\frac{dR}{d\theta} = 0$. So:

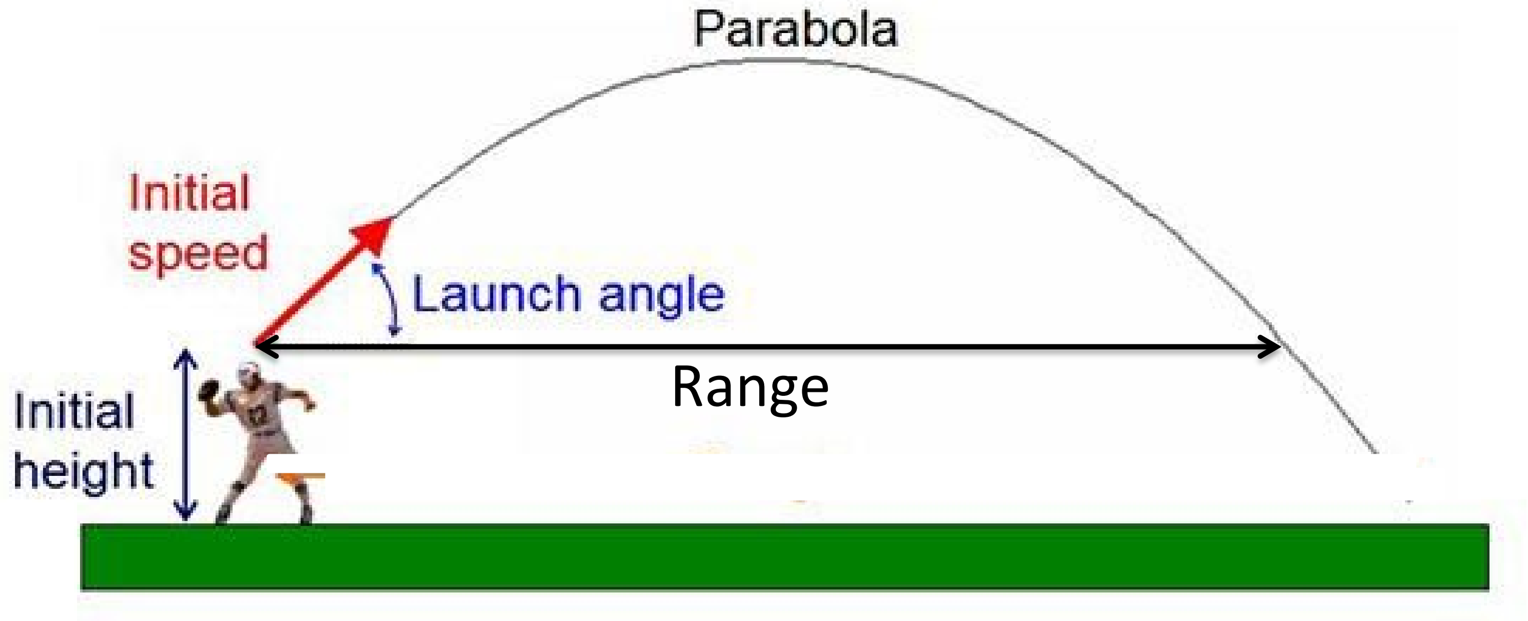
$$0 = \frac{dR}{d\theta} = \frac{v_0^2}{g}\frac{d}{d\theta}(\sin 2\theta\ ) = \frac{v_0^2}{g}(-2\cos 2\theta\ ) = -\frac{2v_0^2}{g}\cos 2\theta$$

Maximum occurs when:

$$\cos 2\theta = 0 \ \rightarrow 2\theta = 90^\circ \rightarrow \boxed{\theta = 45^\circ}$$

Example 4

- A quarterback throws the ball from 35 yards (32 m) the opponent’s goal line. Initial speed of the ball is 20 m/s at a 30° launch angle. The ball leaves his hand at an initial height of 2m.


Parabola
Initial speed
Launch angle
Initial height
Range

Example 4 (continued)

- Given: $v_0 = 20\frac{m}{s},\ \theta = 30°, y_0 = 2\ m,\ x_0 = 0$

(a) How long does it take for the ball to reach the goal line at $x = 32m$?

$$v_{0x} = v_0 \cos\theta = \left(20\frac{m}{s}\right)\cos 30° = 17.32\ m/s$$

$$v_{0y} = v_0 \sin\theta = \left(20\frac{m}{s}\right)\sin 30° = 10\ m/s$$

$$x = v_{0x}t \quad \rightarrow \quad t = \frac{x}{v_{0x}} = \frac{32m}{17.32m/s} = \boxed{1.85\ s}$$

(b) At what height does the ball cross the goal line?

$$\mathrm{y} = y_0 + v_{y0}t - \frac{1}{2}gt^2 = 2 + (10)(1.85) - \frac{1}{2}(9.8)(1.85)^2$$

$$= \boxed{3.73\ m}$$

Continued →

Example 4 (continued)

(c) What is the range? $(v_0 = 20\frac{m}{s}, \theta = 30°)$

$$R = \left(\frac{v_0^2 \sin 2\theta}{g}\right) = \frac{(20\frac{m}{s})^2 \sin 60°}{9.8\, m/s^2} = \boxed{35.35\, m}$$

(d) At what angle must the football be thrown in order to reach receiver 40 m away, i.e. $R = 40\, m$? (Assume v_0 is unchanged and ball is caught at height of 2m.)

$$R = \left(\frac{v_0^2 \sin 2\theta}{g}\right) \quad \rightarrow \quad \sin 2\theta = \frac{gR}{v_0^2} \quad \rightarrow \quad 2\theta = \sin^{-1}\left(\frac{gR}{v_0^2}\right)$$

$$\theta = \frac{1}{2}\sin^{-1}\left(\frac{gR}{v_0^2}\right) = \frac{1}{2}\sin^{-1}\left(\frac{(9.8)(40)}{(20)^2}\right) = \frac{1}{2}(78.52°)$$

$$= \boxed{39.26°}$$

Continued ⇨

Example 4 (continued)

(e) What is the maximum possible range, assuming v_0 remains held constant at $v_0 = 20\frac{m}{s}$?

If is held fixed, the only thing that can be varied to affect the range is the angle. The maximum range occurs at $45°$, so:

$$R_{max} = \left(\frac{v_0^2 \sin 2\theta}{g}\right) = \frac{(20\frac{m}{s})^2 \sin 2(45°)}{9.8\ m/s^2} = \frac{(20\frac{m}{s})^2 \sin(90°)}{9.8\ m/s^2}$$

$$= \boxed{40.82\ m}$$

Note again: the use of the range formula is allowed **only** because we know the initial and final heights are equal.

Chapter 4: Forces

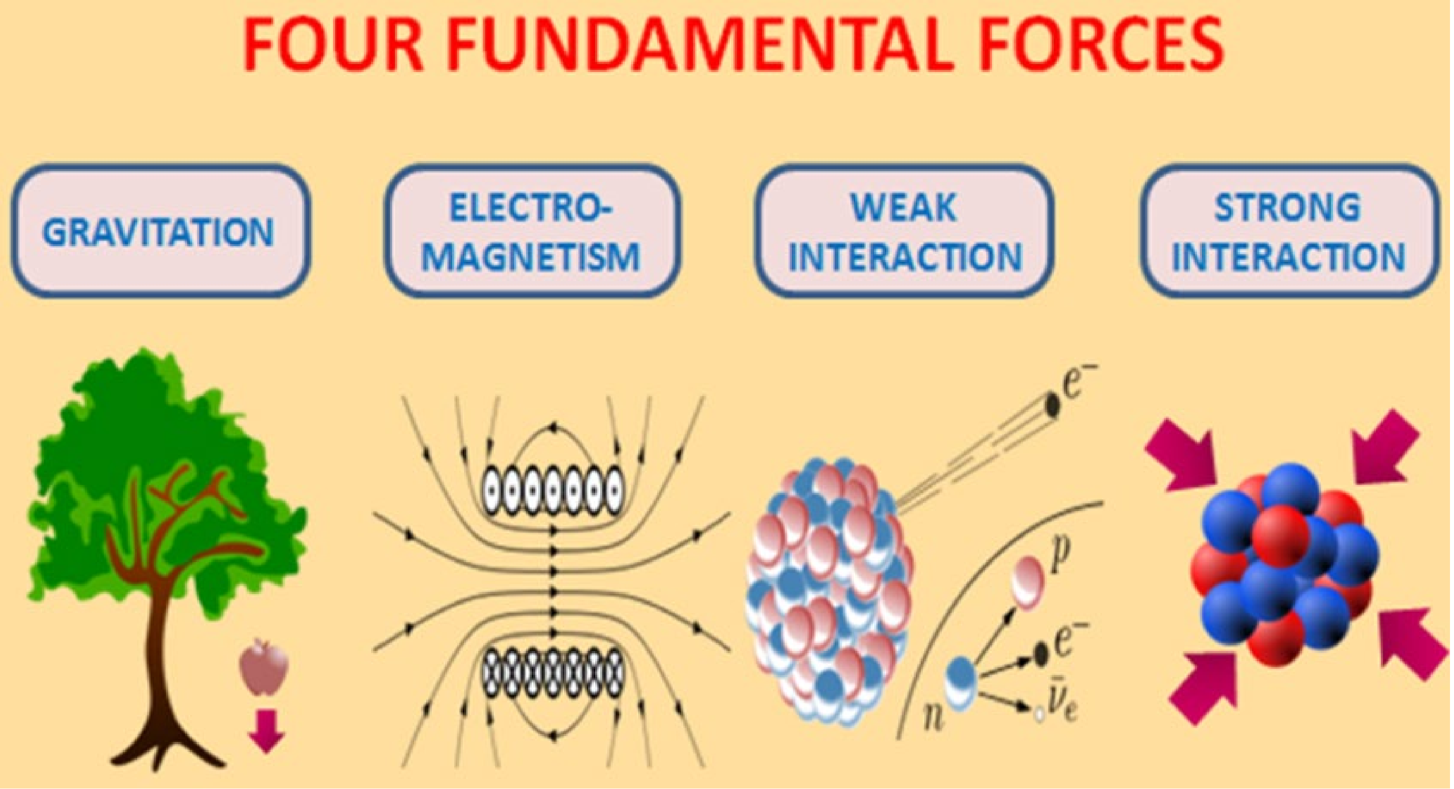
WHEREIN INNOCENT MASSES GET PUSHED AROUND BY BULLYING FORCES

Dynamics

- So far we have been doing **kinematics**: we assumed an acceleration and found the resulting x(t) and v(t).
- Now we want to move into **dynamics**: we want to know how to find the acceleration.
- To do this we need the forces on the system.
- A **force** is a push or pull on an object.
- **Forces are vectors!** They have directions.

Forces

- There are four fundamental forces in nature:


FOUR FUNDAMENTAL FORCES
GRAVITATION
ELECTRO-MAGNETISM
WEAK INTERACTION
STRONG INTERACTION
e^-
p
e^-
$\bar{\nu}_e$
n

Long range Very weak	Long range Strong	Short range Weak	Short range Very strong

Forces

- We won’t deal with nuclear forces. Our main interest will be the long range **electromagnetic** and **gravitational** forces.
- The electromagnetic force will appear in many disguises:
  - Normal forces
  - Friction forces
  - Spring forces
  - Tension
- We won’t see electromagnetism in its bare, undisguised form until next semester.

Forces

- The only non-electromagnetic force we will see is **gravity**. The force of gravity is also called **weight**.
- Don’t confuse weight and mass.
  - Weight is a vector, mass is a scalar.
  - Weight depends on the local gravitational field, while mass is an intrinsic property of an object.
  - The two quantities have different units:
    - Mass is in kilograms
    - Weight has units of force (next slide)

Units of Force

- Our main unit of force will be the **Newton**:

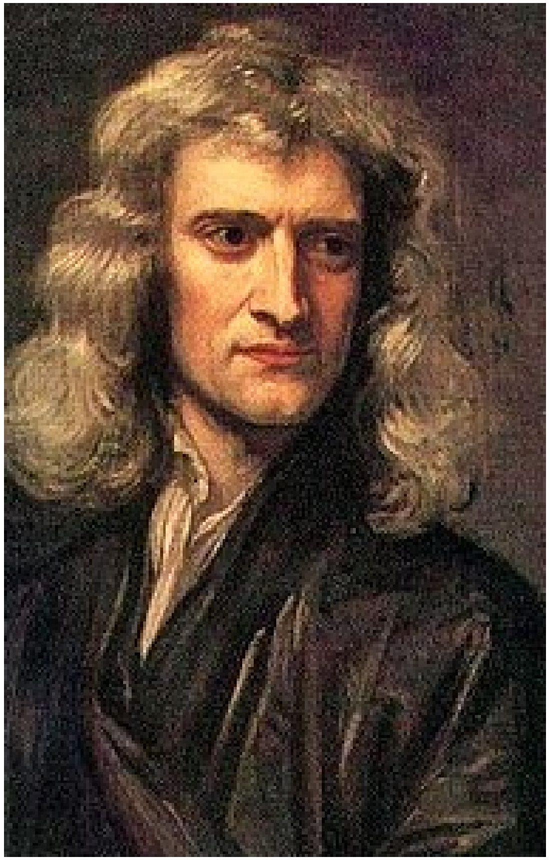
$$\boxed{1\,N = 1\,\frac{kg\,m}{s^2}}$$

- An alternate unit for force and weight is the **pound** (lb):

$$1\,lb = 4.45\,N \quad or \quad 1\,N = .2247\,lb$$

- At the earth's surface, 1 kg of mass will have a weight of 9.8 N = 2.20 lb.

Why Forces?



- The modern concept of force was developed by Galileo and Newton.
- We are interested in forces because they allow us to find accelerations. Once accelerations are known, we can find positions and velocities as functions of time, as in chapters 2 and 3.
- The rules for making use of forces to understand dynamics are embodies in 3 laws developed by Isaac Newton.

Newton’s 1st Law of Motion

1. **Law of Inertia:**

    In the absence of forces, a body at rest tends to remain at rest & every body in motion tends to continue in motion at constant velocity.

This just says that when there is no force, there is no acceleration.

What about when there are forces?

Newton's 2nd Law of Motion

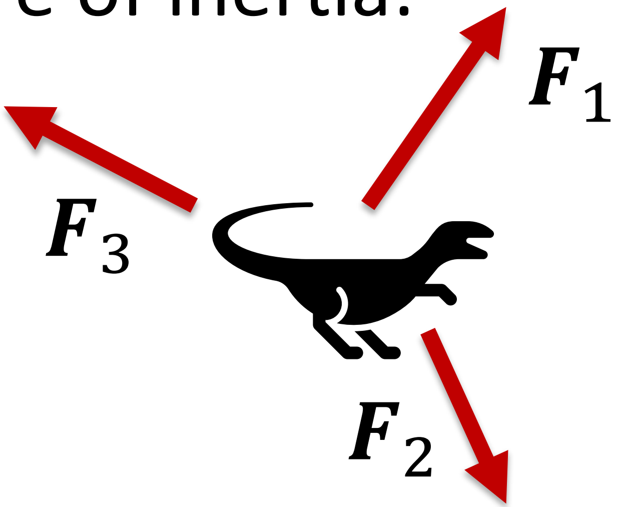
2. Newton's 2nd law:

$$\boxed{\sum \boldsymbol{F} = m\boldsymbol{a}}$$

- The sum is over all the forces on the object.
- The forces are added **as vectors**.
- The mass is acting as a measure of inertia:

  Large *m* means large force is needed to create small ***a***

- The 2nd law is the main focus in this chapter and our main tool for understanding motion

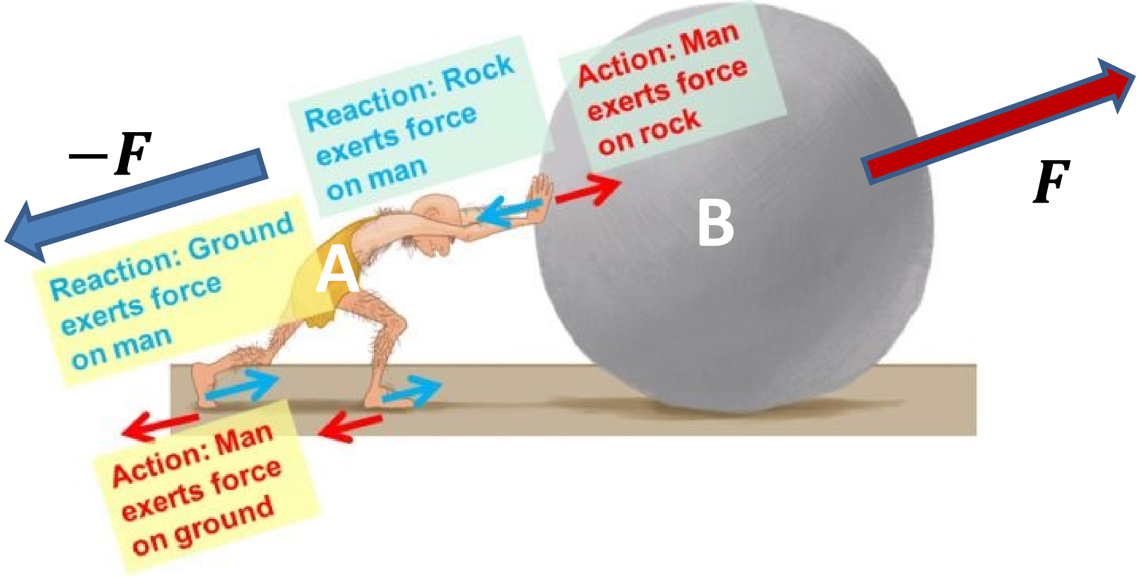


$\boldsymbol{F}_1$
$\boldsymbol{F}_3$
$\boldsymbol{F}_2$

Newton’s 3rd Law of Motion

3. **Law of action and reaction:**

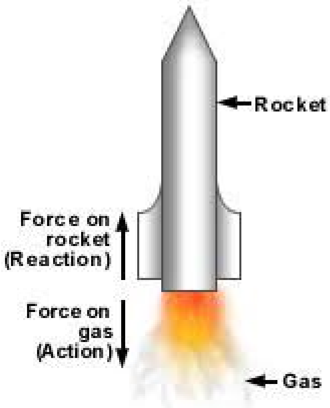
   If object A pushes with force $\boldsymbol{F}$ on object B, then object B pushes back on A with an equal and opposite force, $-\boldsymbol{F}$.

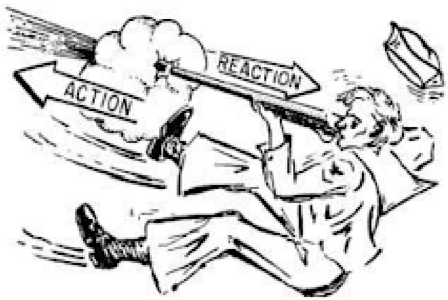


$-\boldsymbol{F}$
Reaction: Rock exerts force on man
Action: Man exerts force on rock
$\boldsymbol{F}$
B
A
Reaction: Ground exerts force on man
Action: Man exerts force on ground

Action and reaction Forces

- Motion occurs because of the 3rd law:
  - Foot pushes against ground, so ground pushes back, propelling you forward
  - Rocket pushes exhaust out back, so exhaust pushes rocket forward


Rocket
Force on rocket (Reaction)
Force on gas (Action)
Gas


REACTION
ACTION

- **Warning:** The action and reaction forces act on different objects. They do not enter the same $F = ma$ equation and **do not cancel**.

Simple force example

- An electric field accelerates an electron by exerting a force of magnitude $3.2 \times 10^{-17}N$ on it. The acceleration occurs over a distance of 0.1 m. If the electron starts from rest, find its final speed.
- Mass of electron: $m = 9.11 \times 10^{-31}kg$
- $F = ma \quad \rightarrow \quad a = \frac{F}{m} = \frac{3.2\times10^{-17}N}{9.11\times10^{-31}kg} = 3.51 \times 10^{13}\,\frac{m}{s^2}$

$$v^2 = v_0^2 + 2a(x - x_0)$$

$$v = \sqrt{v_0^2 + 2a(x - x_0)} = \sqrt{0 + 2(3.51 \times 10^{13})(.1m)}$$

$$= \boxed{2.6 \times 10^6\ \frac{m}{s}}$$

Forces

- The previous example was simple because (i) it had a single force and (ii) all motion was along a single direction.
- For more general problems, we will need to remember that forces are three-dimensional. The vector equation $\Sigma \boldsymbol{F} = m\boldsymbol{a}$ is really shorthand for three equations:
  - $\Sigma F_x = ma_x\ , \quad \Sigma F_y = ma_y\ , \quad \Sigma F_z = ma_z$
- In addition, if there are N objects, there will be n copies of each of these.
- So for N objects in 3 dimensions, there will be 3N equations to solve. Mostly, we stick to 1 or 2 objects in 2 dimensions.

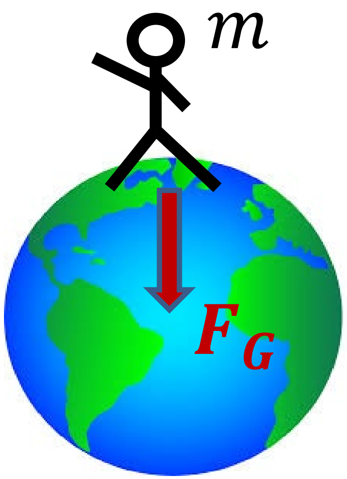
Common Forces: Weight

- **Weight** = Force of gravity = attraction between masses
- For now we assume we are near the surface of the Earth; we'll return to the general case later.
- At Earth's surface, the weight of an object is due to the pull of the Earth's mass on the object's mass.
- As long as the altitude of the object doesn't change too much, the weight is constant.

Common Forces: Weight

- Let m be the mass of the object.
- Then the force of gravity is:

$$F_G = mg \qquad (magnitude)$$
$$F_{Gx} = 0 \qquad (x\ component)$$
$$F_{Gy} = -mg \qquad (y\ component)$$



- As vector: $\boxed{\boldsymbol{F_G} = -mg\,\boldsymbol{k}}$

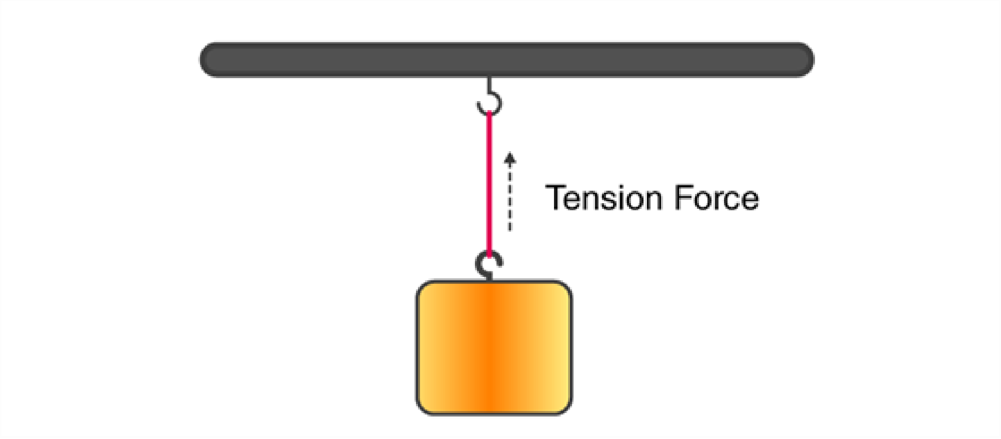
- The force is vertically **downward**, toward the center of the earth

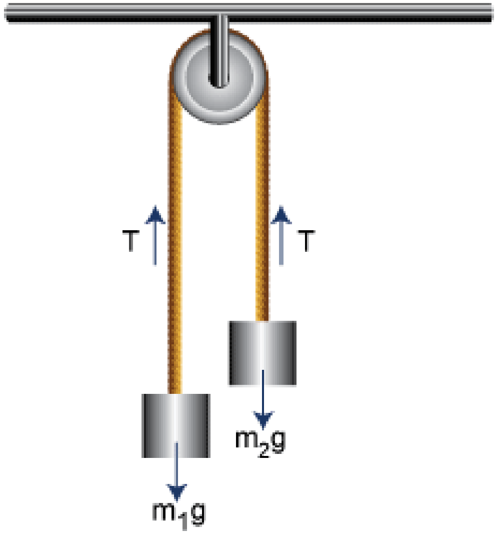
A Note on Mass

- Mass plays two roles in physics.
- In $\Sigma \boldsymbol{F} = m\boldsymbol{a}$, it measures inertia (resistance to acceleration). (**Inertial mass**)
- In $F_G = mg$, it plays a coupling constant (measure of how strongly gravity couples to the object). (**Gravitational mass**)
- There is no known reason why these two masses should be equal, but they always are.
- This is called the equivalence principle, and was one of Einstein's starting points in constructing General Relativity.

Common Forces: Tension

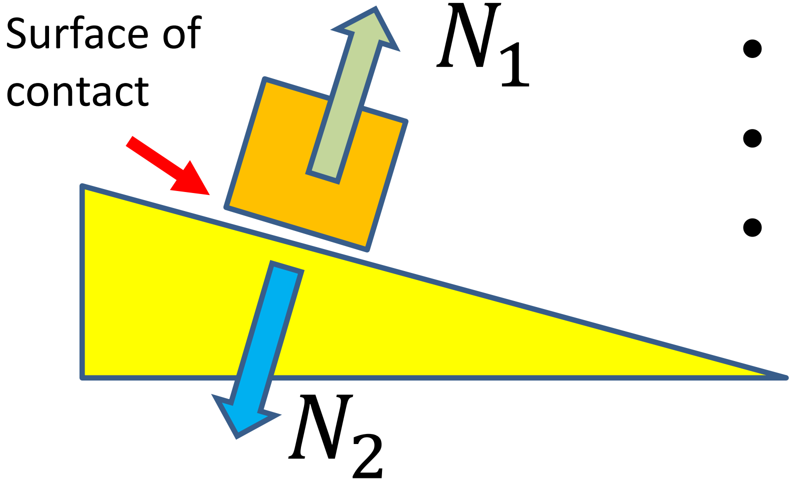
- **Tension: T** = Pull exerted by rope, wire, cable, …
- Direction: parallel to rope, away from object
- To find magnitude: must solve $\Sigma\,\boldsymbol{F} = m\boldsymbol{a}$
- If mass of rope is negligible, its magnitude is constant throughout the rope.





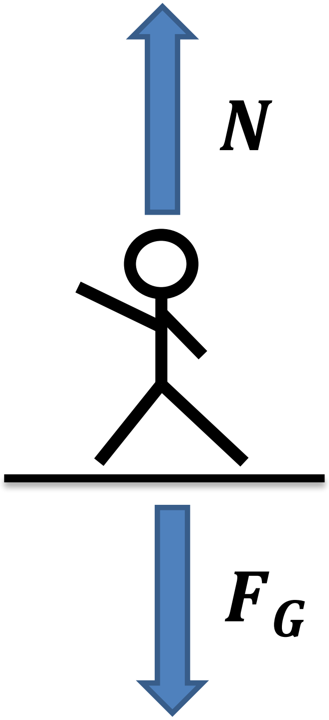
Common Forces: Normal Force

- **Normal force: N** = Push exerted by objects in contact with each other
- Direction: perpendicular to surface of contact, pushing objects away from each other
- To find magnitude: must solve $\Sigma \boldsymbol{F} = m\boldsymbol{a}$



- N_1 is the normal force on block
- N_2 is the normal for on ramp
- They are equal and opposite forces (3rd law)

Normal Force Example



- Object on level, stationary surface.
- No acceleration, so:

$$\sum \boldsymbol{F} = m\boldsymbol{a} \quad \rightarrow \quad N - mg = 0$$

- So: $N = mg$

- Note:
  - We only needed y-components in this case, since x-components all vanish.
  - Do not automatically assume that $N = mg$. It is **not** always true (see examples below).

More Common Forces

- Additional forces that we will see later:
  - **Friction:** We'll come back to this in the next chapter.
  - **Hooke's law (elastic) forces:** ch. 5
  - **Magnetic and electrostatic forces:** next semester

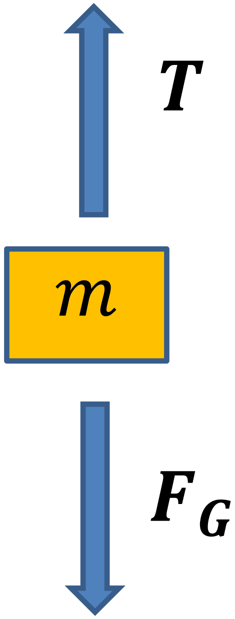
- Once again: elastic, friction, normal, and tension forces are all ultimately electromagnetic in origin.

Solving Force Problems

- **Goal**: Given forces, find acceleration (or vice versa)
- **Basic strategy:**
  - Draw force diagram. There is usually a force from each object in contact with the object of interest, plus gravity.
  - Draw a separate diagram for each object.
  - Give each force a name, and make its direction clear.
  - Break each force into components
  - Write down the $\Sigma \boldsymbol{F} = m\boldsymbol{a}$ equations for each component (x,y,z) and for each object
  - Do the algebra to solve the resulting equations to find the unknowns

Example 1 (1D, 1 object)

- A block is being pulled upward with a rope. Find the tension required to produce an acceleration of 3 m/s^2. Assume m=2 kg.


$\boldsymbol{T}$
m
$\boldsymbol{F_G}$

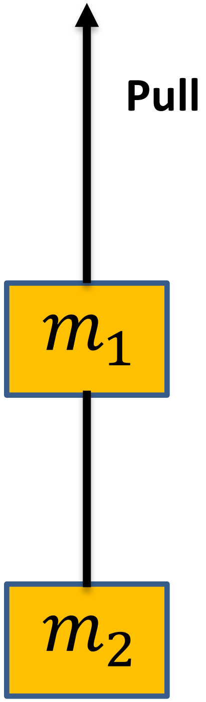
$$\Sigma F_y = ma_y \quad \rightarrow \quad T - mg = ma$$

$$T = ma + mg = m(a+g)$$

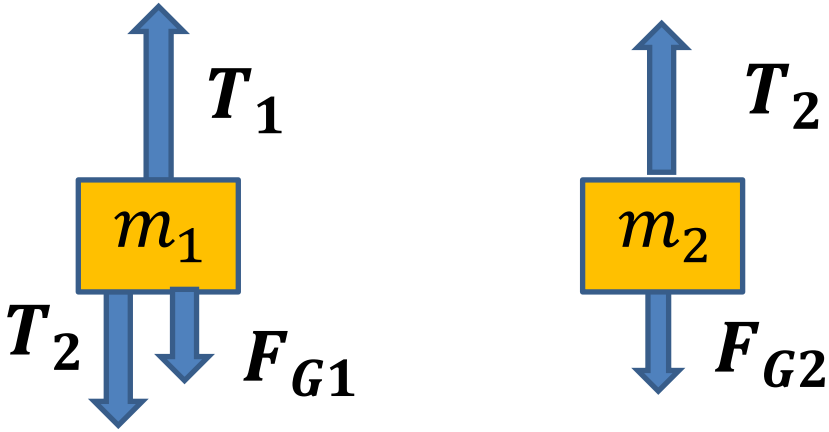
$$= (2\,kg)(3+9.8) = \boxed{25.6\,N}$$

Example 2 (1D, 2 objects)

- Now *two* blocks are pulled upward by rope. Find the tension in each rope. Assume $m_1=m_2=2$ kg, $a = 3 m/s^2$.



Two objects means drawing two force diagrams:


$\boldsymbol{T_1}$
m_1
$\boldsymbol{T_2}$
$\boldsymbol{F_{G1}}$
$\boldsymbol{T_2}$
m_2
$\boldsymbol{F_{G2}}$

There are only y-components. Write down $\Sigma F_y = ma_y$ equations for each mass

Example 2 (1D, 2 objects)

- **Bottom Block:**

$$\Sigma F_y = ma_y \quad \rightarrow \quad T_2 - m_2 g = m_2 a$$

$$T_2 = m_2 a + m_2 g = m_2(a + g)$$

$$= (2\ kg)(3 + 9.8) = \boxed{25.6\ N}$$

Same a: rope forces $a_1 = a_2$

- **Top Block:**

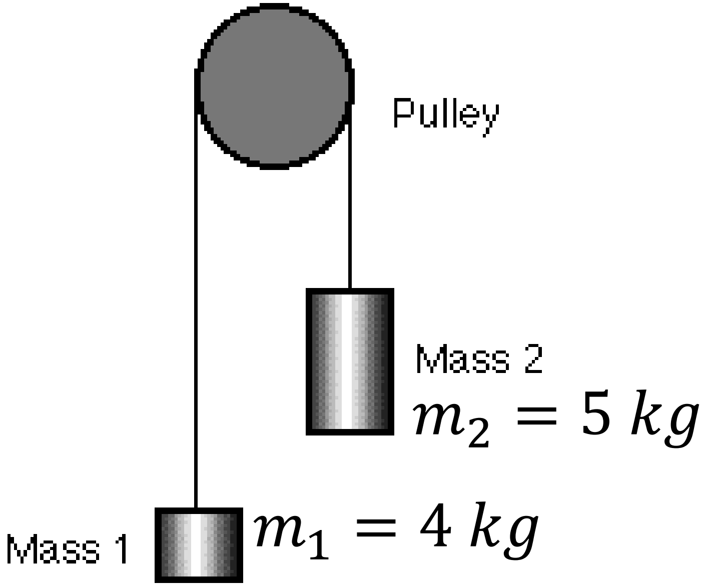
$$\Sigma F_y = ma_y \quad \rightarrow \quad T_1 - T_2 - m_1 g = m_1 a$$

$$T_1 = m_1(a + g) + T_2$$

$$= (2\ kg)(3 + 9.8) + 25.6 = \boxed{51.2\ N}$$

Example 3 (1D, 2 objects)

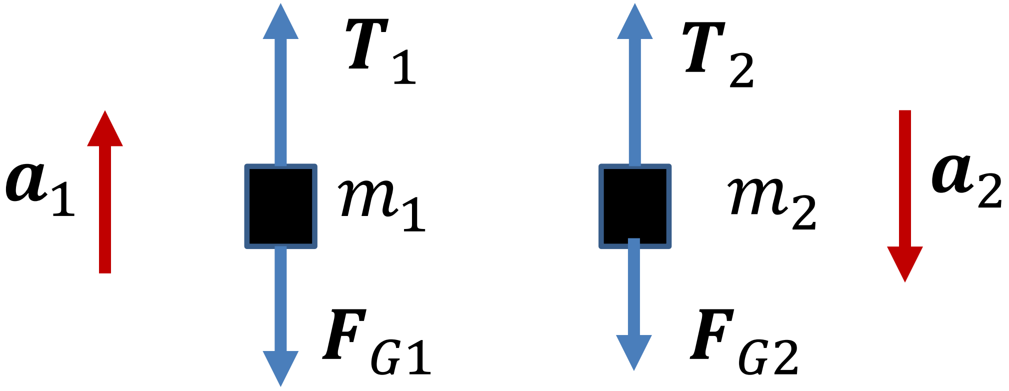
- Two blocks of masses and are connected to a massless rope which runs over a pulley. When released, find acceleration of each block.


Pulley
Mass 2
Mass 1

$m_2 = 5\ kg$

$m_1 = 4\ kg$

Force diagrams:


$\boldsymbol{T}_1$
$\boldsymbol{T}_2$
$\boldsymbol{a}_1$
m_1
m_2
$\boldsymbol{a}_2$
$\boldsymbol{F}_{G1}$
$\boldsymbol{F}_{G2}$

Now the accelerations are equal and opposite: $a_2 = -a_1$.

Only y-components are relevant.

Example 3 (continued)

- First block: $T_1 - m_1 g = m_1\, a_1$
- Second block: $T_2 - m_2 g = m_2 a_2$
- But $T_1 = T_2$ and $a_1 = -a_2$, so:

$$\left.\begin{aligned} T_1 - m_1 g &= m_1\, a_1 \\ T_1 - m_2 g &= -m_2\, a_1 \end{aligned}\right\} \Rightarrow T_1 = m_1(a_1 + g)$$

$$\Downarrow$$

$$m_1\,(a_1 + g) - m_2 g = -m_2 a_1$$

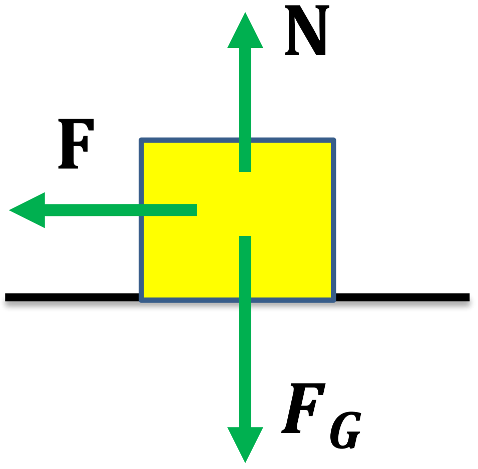
$$(m_1 - m_2)g = -(m_1 + m_2)a_1$$

$$a_1 = \left(\frac{m_2 - m_1}{m_2 + m_1}\right) g = \left(\frac{5-4}{5+4}\right)(9.8) = \boxed{1.09\; m/s^2}$$

$$a_1 = -a_2 = \boxed{-1.09\; m/s^2}$$

Example 4 (2D)

- A box with mass $m = 2\ kg$ is lying on a table. Someone pushes to the left with $\mathbf{F} = 10\ N$ force. Find normal force and acceleration.



Vertical forces

$$F_G = mg = (2kg)\left(9.8\frac{m}{s^2}\right) = 19.6\ N$$

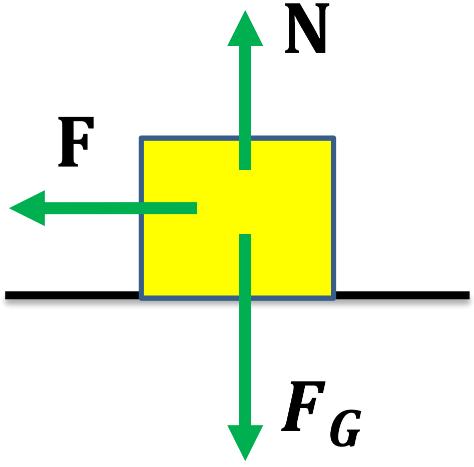
$$\Sigma F_y = ma_y \ \rightarrow\ N - mg = 0$$

$$N = mg = \boxed{19.6\ N}$$ (Upward)

Now look at horizontal direction →

Example 4 (continued)

- **Horizontal forces**



$$\Sigma F_x = ma_x \quad \rightarrow \quad -F = ma_x$$

$$a_x = -\frac{F}{m} = -\frac{10\ N}{2\ kg} = -5\frac{m}{s^2}$$

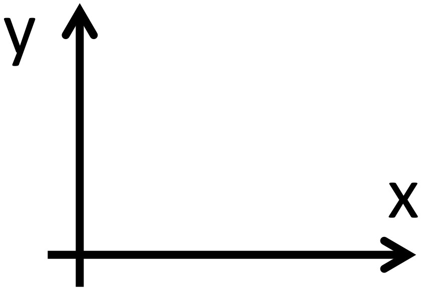
$$\boxed{a_x = -5\frac{m}{s^2}, \quad a_y = 0}$$

(To Left)

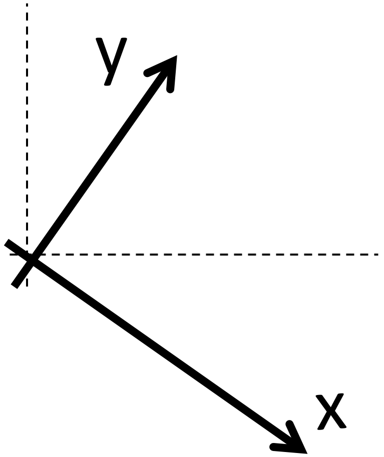
Note: although this example involved two dimensions, it was simply because the forces were all either vertical or horizontal: there were none at an angle. This usually won't be the case (see next few examples).

Digression: Tilted axes

- Normally we take the x and y axes to be vertical and horizontal:



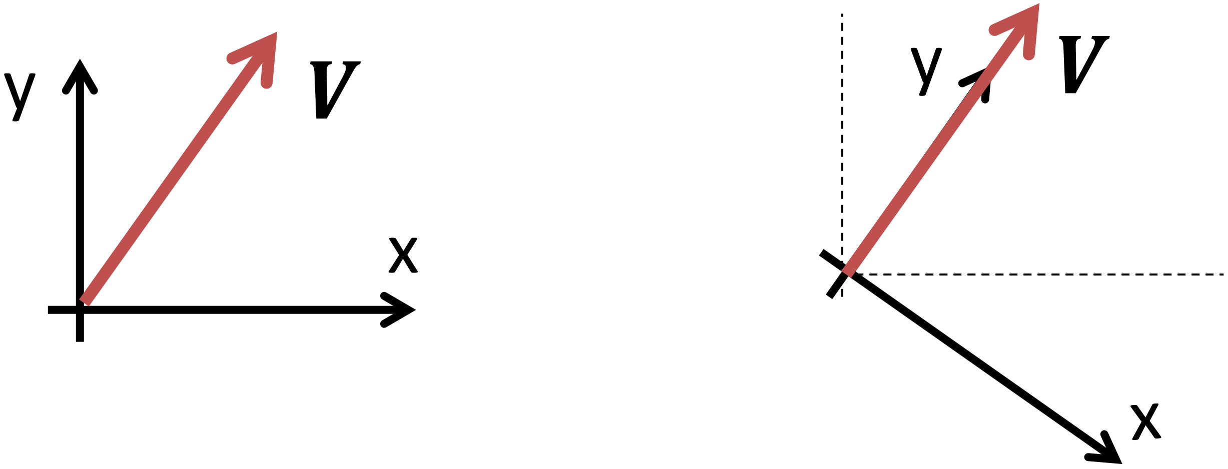
- But we don't have to.
- We can take them in any directions we want:



- We can use this to our advantage to simplify problems.

Digression: Tilted axes

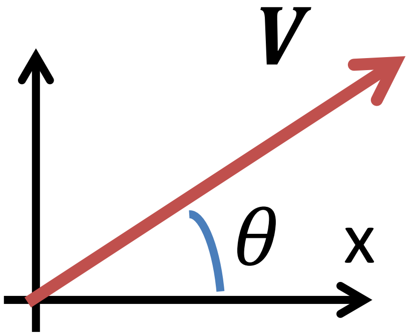
- The vector $\boldsymbol{V}$ below has two components in original coordinate system (left), but only one nonzero component in the tilted system (right).



- By tilting our coordinate system so that as many vectors as possible are along the axes, we can minimize the algebra by making many of the vector components vanish.

Digression: Tilted axes

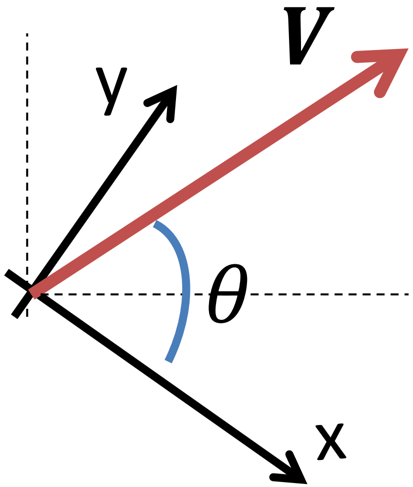
- When tilting axes, **be careful**: the angle used to find components is the angle from whatever direction you chose for the positive x-axis.



$\boldsymbol{V}$
y
θ
x

$$V_x = V cos\theta$$

$$V_y = V sin\theta$$

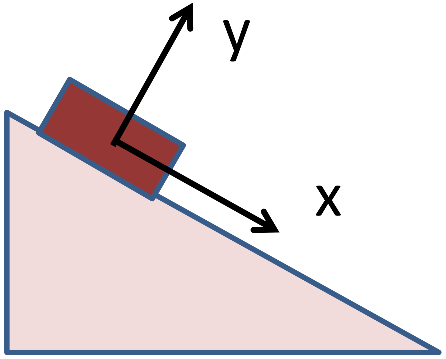


$\boldsymbol{V}$
y
θ
x

The components and the angle will be different in each of these coordinate systems, but the magnitude will be the same.

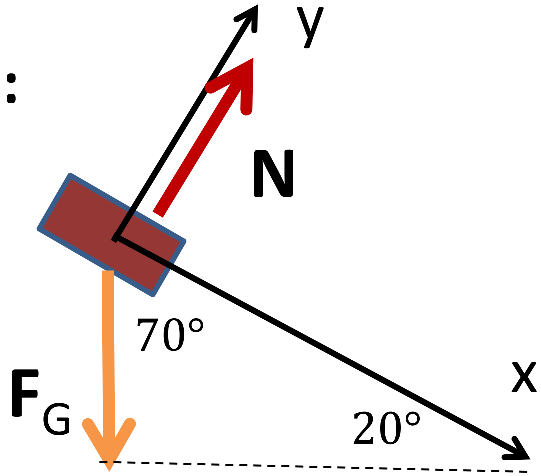
Example 5

- A 2 kg box sits on a frictionless surface, tilted at a 20° angle. Find normal force on the box, and its acceleration.



Take x and y axes to be parallel and perpendicular to the ramp, as shown. Then: $a_y = 0\,, a_x = a$

Force diagram:



Notice that the angles from the positive x axis for the two forces are:

$$\theta_N = 90°$$
$$\theta_G = 70°$$

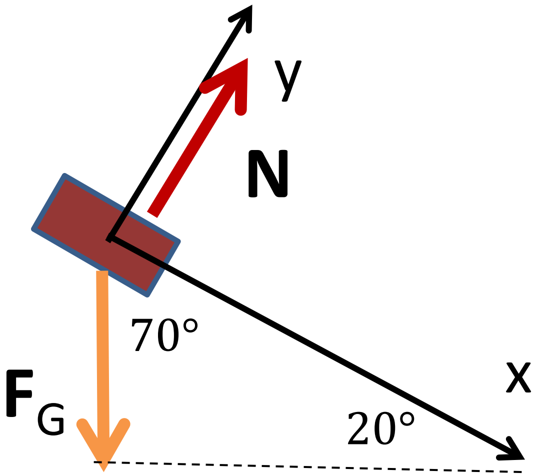
Find components

Example 5 (continued)

- Find components:

$N_x = 0 \qquad F_{Gx} = mgcos(-70°) = +mgcos(70°)$

$N_y = N \qquad F_{Gy} = mgsin(-70°) = -mgsin(70°)$



- Solve $\Sigma\, \boldsymbol{F} = m\boldsymbol{a}$ equations:

- x-direction:

$$\Sigma\, F_x = ma_x \;\; \rightarrow \;\; N_x + F_{Gx} = ma_x \;\; \rightarrow \;\; 0 + mg\cos\theta = ma$$

$$\text{a} = g\cos\theta = (9.8)\cos 70° = \boxed{3.35\, m/s^2}$$

- y-direction:

$$\Sigma\, F_y = ma_y \;\; \rightarrow \;\; N_y + F_{Gy} = ma_y \;\; \rightarrow \;\; N - mg\sin 70° = 0$$

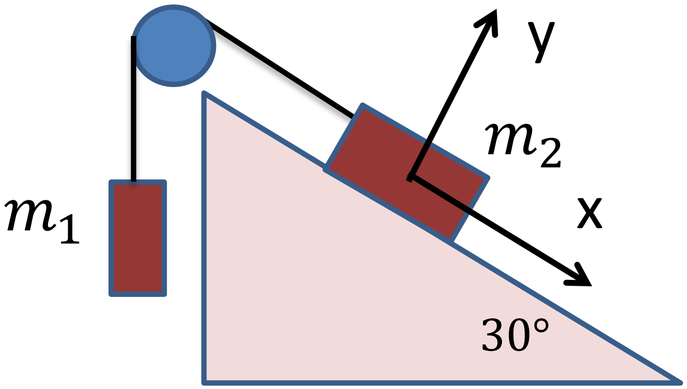
$$\text{N} = mg\sin\theta = (2)(9.8)\sin 70° = \boxed{18.42\, N}$$

(Note: $\boldsymbol{N \neq mg}$!)

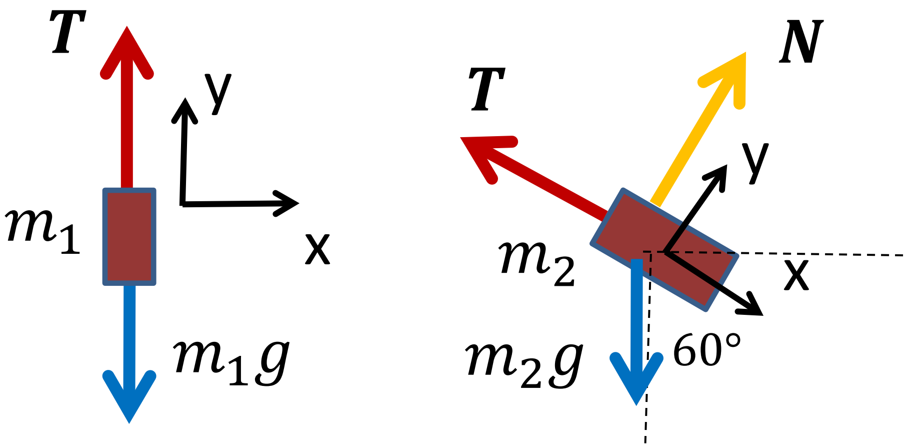
Example 6

- Two masses are connected by a rope hanging over a frictionless pulley. Assume $m_2 = 4m_1$.

(a) Find acceleration. Is it up or down the ramp?



Draw force diagram for each block.


$\boldsymbol{T}$
y
x
m_1
$m_1 g$
$\boldsymbol{N}$
$\boldsymbol{T}$
y
m_2
x
$m_2 g$
$60°$

As long as we are careful, we can use different coordinate system for each block. Use axes that make components of each one simplest.

Find components

Example 6 (continued)

- Note that the accelerations are related again:

$$a_{1x} = a_{2y} = 0, \qquad a_{1y} = a_{2x} \ \ (= a)$$

- **Block 1**: Only y-components are nonzero.

$$\Sigma\, F_y = ma_y \quad \rightarrow \quad \mathrm{T} - m_1 g = m_1 a$$

$$\mathrm{T} = m_1(g + a)$$ Ⓐ

- **Block 2:** With axes chosen, only x-component of a is nonzero

$$\Sigma\, F_x = ma_x \quad \rightarrow \quad -\mathrm{T} + m_2 g \cos 60° = m_2 a$$

$$\mathrm{T} = m_2(g \cos 60° - a)$$ Ⓑ

Example 6 (continued)

- Combine Ⓐ and Ⓑ to eliminate T.

$$m_1(g+a) = m_2(g\cos 60° - a)$$

$$(m_2 + m_1)a = (m_2\cos 60° - m_1)g$$

$$a = \frac{(m_2\cos 60° - m_1)}{(m_1+m_2)}g = \frac{(4\cos 60° - 1)}{(1+4)}\, g = \boxed{1.96\,\frac{m}{s^2}}$$

(Positive : **Down** the ramp)

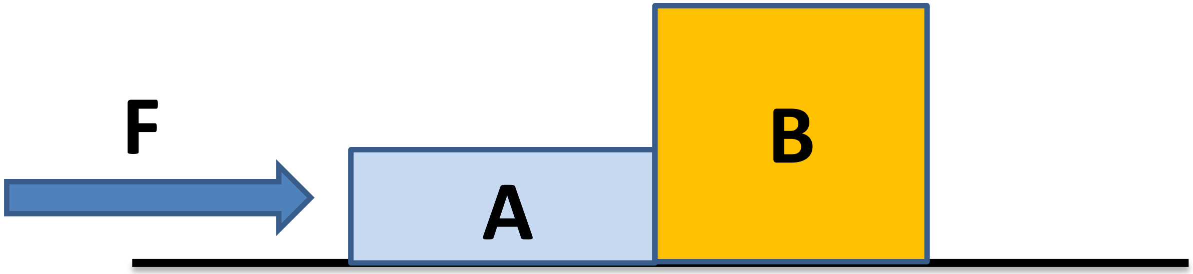
(b) Normal force?

From y-components of block 2:

$$N = m_2 g\sin\theta = 4(9.8)\ \sin 60° = \boxed{34\ N}$$

Extra Example

- A force F is exerted on a pair of blocks sitting on the floor: $F = 20\ N, m_A = 4\ kg, m_B = 6\ kg$.



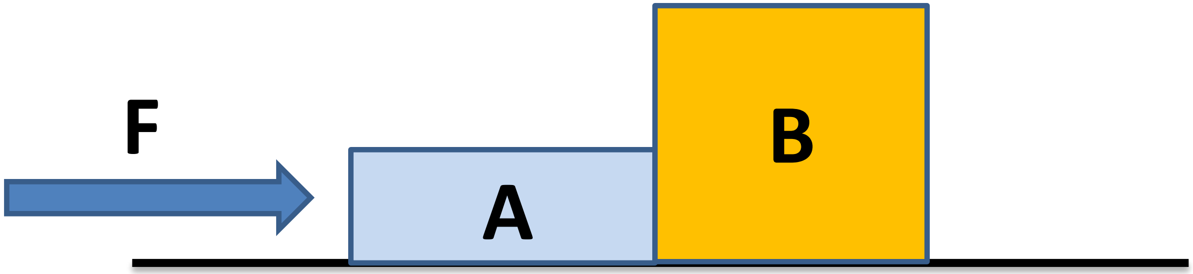
- Find:

(a) The acceleration of the blocks

(b) The magnitude of the normal force each block exerts on the other.

Extra Example

- A force F is exerted on a pair of blocks sitting on the floor: $F = 20\,N, m_A = 4\,kg, m_B = 6\,kg.$



- Find:

(a) The acceleration of the blocks

(b) The magnitude of the normal force each block exerts on the other.

Answers: (a) 2 m/s^2 , (b) 12N

Chapter 5: Applications of forces

IN WHICH MANY MORE THINGS GET PUSHED AROUND

This Chapter

- This chapter looks at more applications of the material from chapter 5.
- It focuses primarily on applications involving two new ingredients:
  - Friction
  - Circular motion

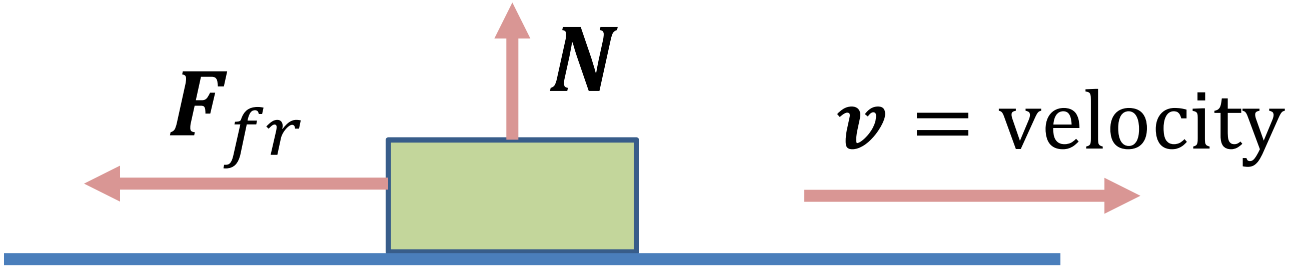
Friction

- **Friction** is a force that resists sliding motion of one object over the surface of another.
- We will denote the friction force $\boldsymbol{F}_{fr}$. Its direction is always parallel to the surface of contact.
- Friction is ultimately due to electromagnetic forces between microscopic bumps and ridges in the surfaces.
- There are two types of friction:
  - Kinetic
  - Static

Kinetic Friction

- **Kinetic friction** acts on moving bodies, & tries to bring them to rest
- Direction: Opposite to object’s velocity
- Magnitude: $\boxed{F_{fr} = \mu_K N}$
- μ_K = **Coefficient of kinetic friction**, gives measure of roughness of surface

(μ is the Greek letter mu)

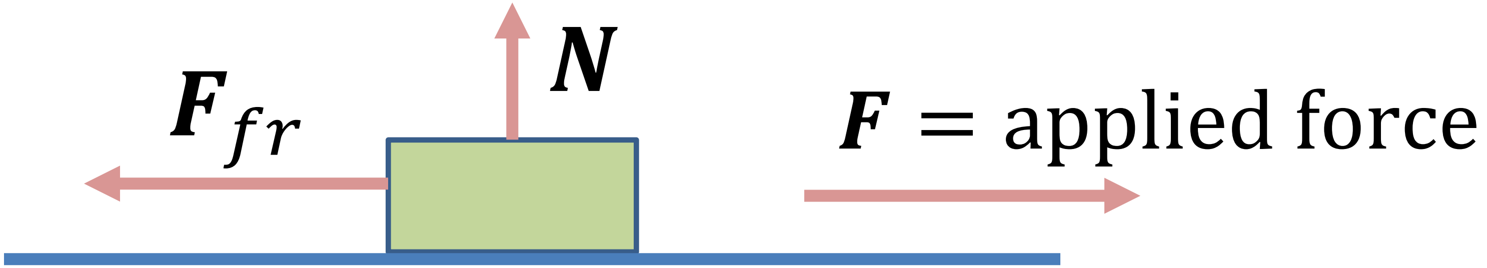

$\boldsymbol{F}_{fr}$
$\boldsymbol{N}$
$\boldsymbol{v}$ = velocity

Kinetic Friction

- Coefficient of kinetic friction:
  - μ_K has no units; it is a pure number
  - μ_K must be between 0 and 1: $0 \leq \mu_K \leq 1$
- Note that the equation $F_{fr} = \mu_K N$ is a relation between the **magnitudes** of the friction and normal forces $F_{fr} = \mu_K N$
- The force **vectors** are in different directions (perpendicular to each other)

Static Friction

- **Static friction** acts on body at rest, & tries to keep it at rest when you push on it
- Direction: Opposite to applied force (push)
- Magnitude: $\boxed{F_{fr} \leq \mu_S N}$
- μ_S = **Coefficient of static friction**, gives measure of roughness of surface


$\boldsymbol{F}_{fr}$
$\boldsymbol{N}$
$\boldsymbol{F} = \text{applied force}$

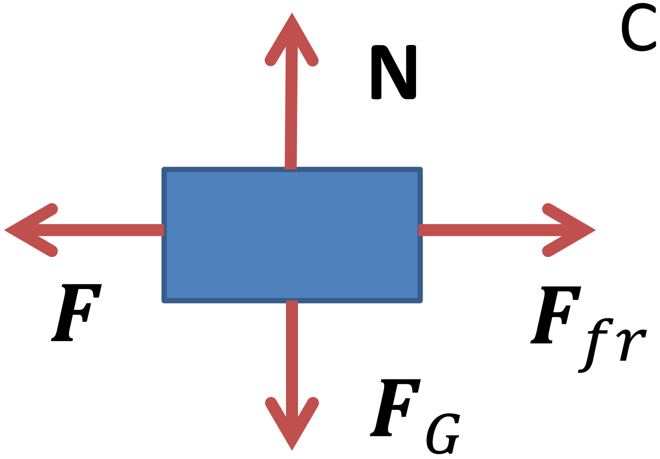
Static Friction

- Coefficient of static friction:
  - μ_S has no units; it is a pure number
  - μ_S must be between 0 and 1: $0 \leq \mu_S \leq 1$
  - It is always larger than the kinetic coefficient: $\mu_S \geq \mu_K$

Example 1

- A block of mass m=10 kg is sitting on a horizontal tabletop. The surface of contact has $\mu_K = 0.2$ and $\mu_S = 0.3$.

(a) If a force of magnitude F = 3 N is applied toward the left, find the friction force and the acceleration of the block.


N
$\boldsymbol{F}$
$\boldsymbol{F}_{fr}$
$\boldsymbol{F}_G$

Components:

$$F_y = 0\,, \qquad F_x = -3\text{ N}$$

$$F_{Gy} = -mg = -98N\,, \qquad F_{Gx} = 0$$

$$N_y = +mg = +98N\,, \qquad N_x = 0$$

What about $\boldsymbol{F}_{fr}$?

Example 1 (continued)

- Is friction static or kinetic? Check:
  - Maximum possible static friction:

    $$F_{fr,max} = \mu_S N = (.3)(98\ N) = 29.4\ N$$

  - Applied force: $F = 3\ N$
  - Since $F < F_{fr,max}$, friction is capable of cancelling F. So the friction is **static.**
- Conclude: $\boxed{a = 0}$ and $\boxed{F_{fr} = F = 3\ N}$

It can't be static if it is accelerating

For a to be zero, the horizontal forces must cancel.

Example 1 (continued)

(b) Now the applied force is increased to F=35 N. Find the new acceleration.

- Now F exceeds the maximum static friction, so the object slides and the friction is kinetic:

$$F_{fr} = \mu_K N = (.2)(98\ N) = \boxed{19.6\ N}$$

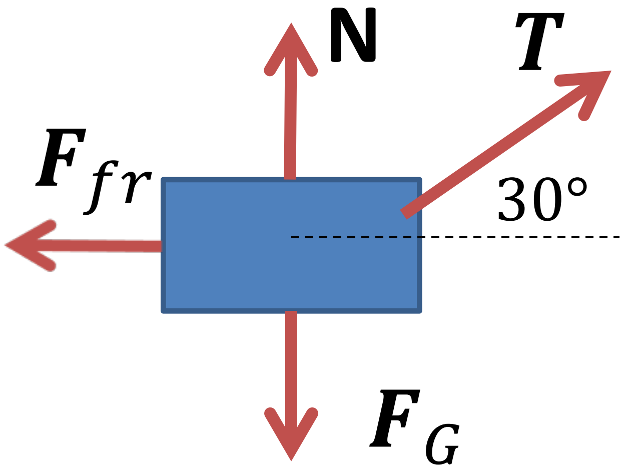
$$\Sigma F_x = ma_x \quad \rightarrow \quad F_{fr} - F = ma$$

$$a = \frac{F_{fr} - F}{m} = \frac{19.6 - 35}{10} = \boxed{-1.54\ m/s^2}$$

To left

Example 2

- A crate weighing 100N is being dragged across the floor by means of a rope at 30° from the horizontal. If the tension in the rope is 40 N and the crate's velocity is constant, find F_{fr} and μ_K.

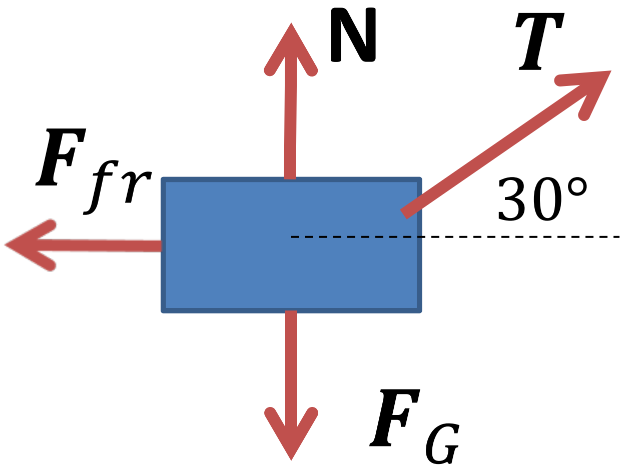


N
$\boldsymbol{T}$
$\boldsymbol{F}_{fr}$
30°
$\boldsymbol{F}_G$

- Since v is constant, we know that $a_x = a_y = 0$
- Since $v \neq 0$, we know the friction is kinetic

Example 2 (continued)

- A crate weighing 100N is being dragged across the floor by means of a rope at 30° from the horizontal. If the tension in the rope is 40 N and the crate's velocity is constant, find F_{fr} and μ_K.



Components:

T= 40 N

$$T_x = T\cos 30° \;, \qquad T_y = T\sin 30°$$

$$F_{Gx} = 0, \qquad F_{Gy} = -mg = -100\;N$$

$$N_x = 0, \qquad N_y = +N$$

$$F_{frx} = -\,\mu_K N, \qquad F_{fry} = 0$$

Example 2 (continued)

- $\Sigma F_y = ma_y \rightarrow \quad N - F_G + T\sin 30° = 0$
- $\Sigma F_x = ma_x \rightarrow \quad T\cos 30° - \mu_K N = 0$
- From the first equation:

$$N = F_G - T\sin 30° = 100 - 40\sin 30° = \boxed{80\ N}$$

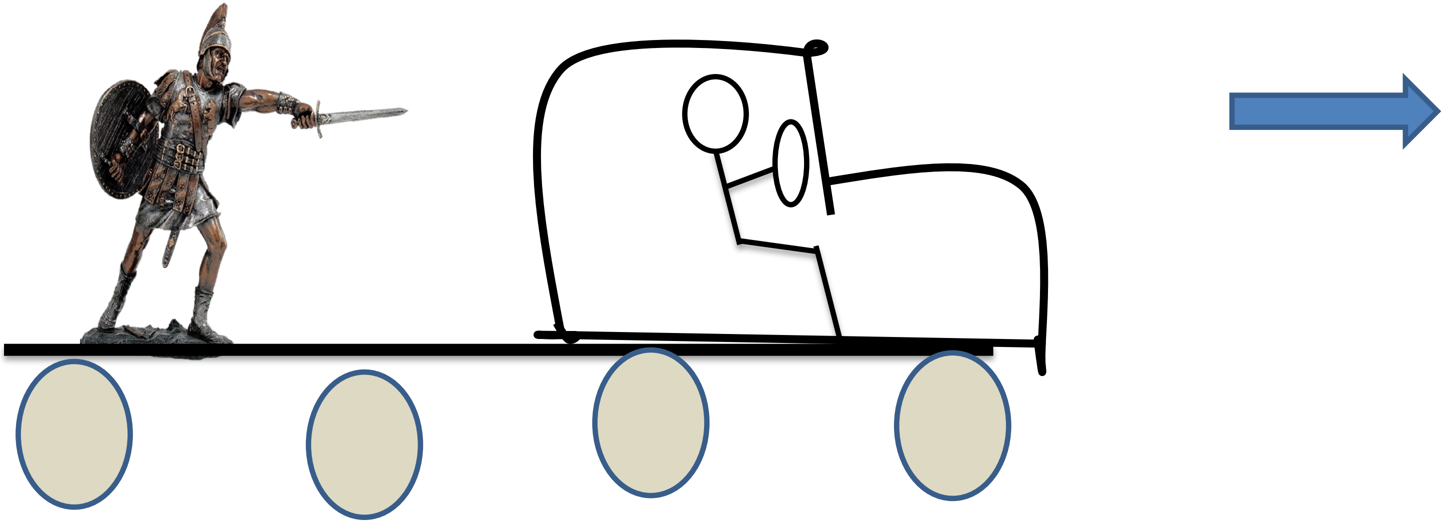
- From the second equation:

$$\mu_K = \frac{T\cos 30°}{N} = \frac{40\ N\cos 30°}{80\ N} = \boxed{.433}$$

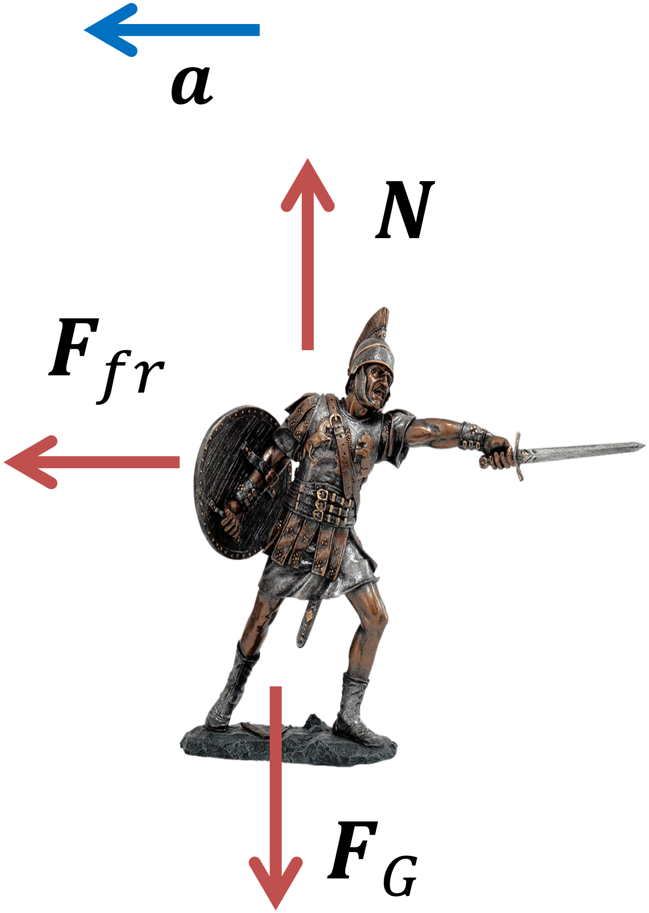
- $F_{fr} = \mu_K N = (.433)(80) = \boxed{34.6\ N}$

Example 3

- An iron statue of mass m=2000kg is on the back of a truck moving at constants speed. The driver hits the brakes as he sees a red light, causing deceleration of 5 m/s^2 . The coefficient of static friction with the truck bed is μ_S=.6. Will the cargo remain in place or will it slide forward?



Example 3 (continued)


$\boldsymbol{a}$
$\boldsymbol{N}$
$\boldsymbol{F}_{fr}$
$\boldsymbol{F}_G$

- We want to know if the static friction is large enough to cause acceleration $a_x = -5\ {}^{m}/_{s^2}$

$$\Sigma F_y = ma_y \quad \rightarrow \quad N - F_G = 0$$

$$N = F_G = mg = (2000)(9.8)$$

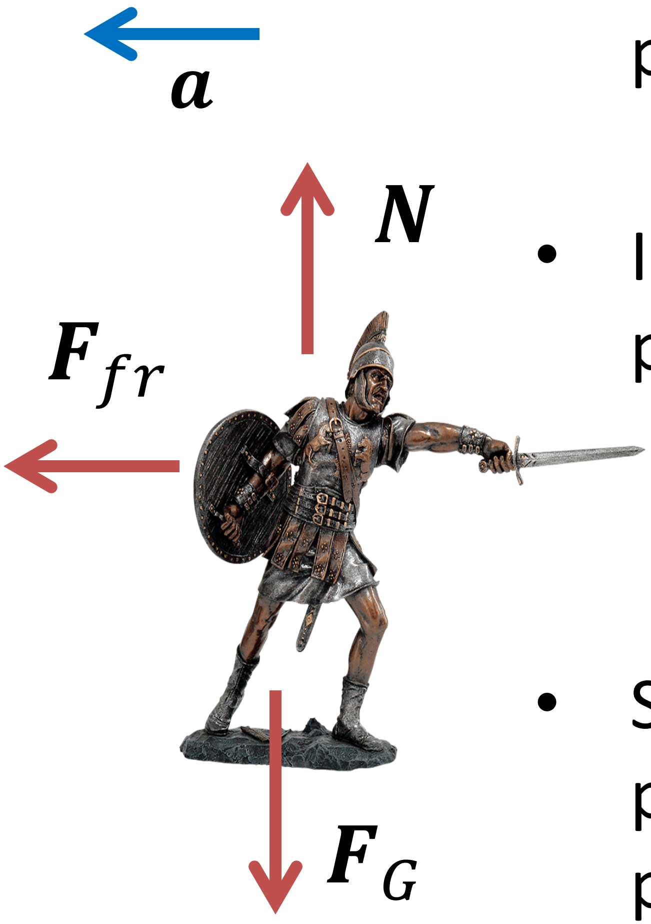
$$N = 19600\ N$$

$$\Sigma F_x = ma_x \quad \rightarrow \quad -F_{fr} = ma_x$$

$$F_{fr} = -ma_x = -(2000)(-5)$$

$$F_{fr} = 10{,}000\ N$$

Example 3 (continued)


$\boldsymbol{a}$
$\boldsymbol{N}$
$\boldsymbol{F}_{fr}$
$\boldsymbol{F}_G$

- The static friction we need to prevent sliding is:

$$F_{fr,needed} = 10{,}000\ N$$

- Is this large a static friction force possible?

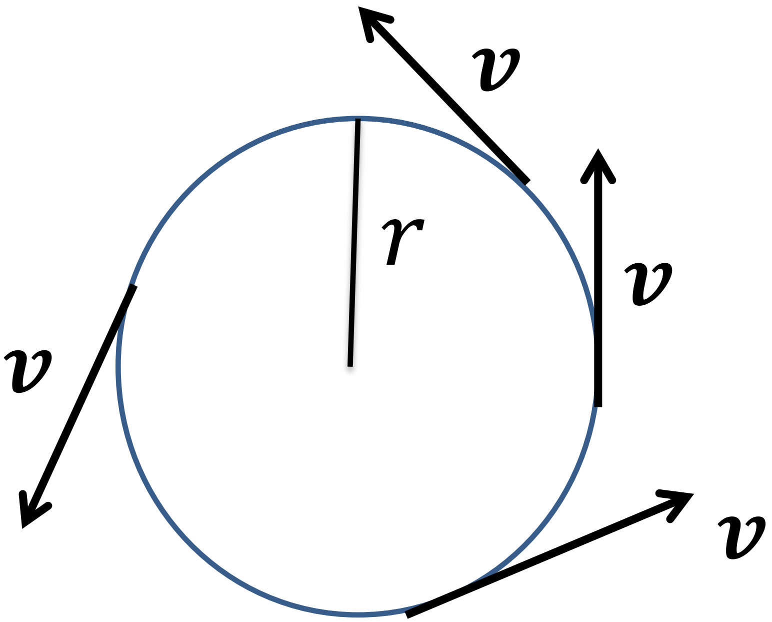
$$F_{fr} \leq \mu_S N = (.6)(19600)$$

$$F_{fr,max} = 11{,}760\ N$$

- So, the friction is capable of producing the force needed to prevent sliding.
- **No sliding:** Driver does not become shish kebob

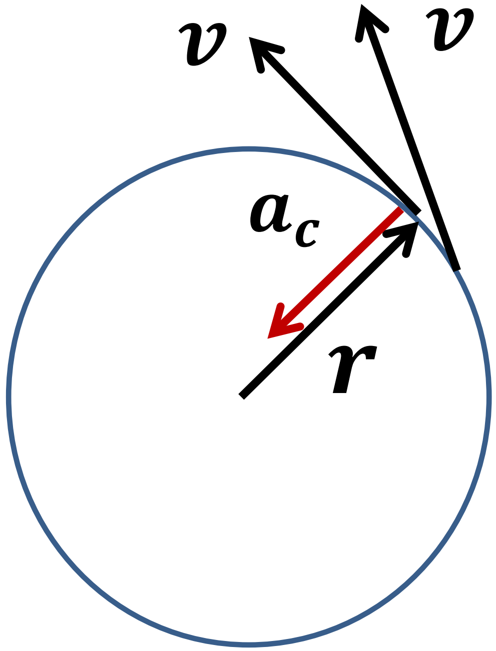
Uniform Circular Motion

- **Uniform circular motion**: motion in circle of radius r, at constant speed.
- **Note**: speed is constant, but velocity is not (since direction is constantly changing).
- $\boldsymbol{v}$ rotates over time, remaining tangent to circle


$\boldsymbol{v}$
r
$\boldsymbol{v}$
$\boldsymbol{v}$
$\boldsymbol{v}$

Uniform Circular Motion

- If there was no acceleration, the object would continue in a straight line (Newton's 3rd law).
- To bend direction of $\boldsymbol{v}$, there must be an acceleration **inward**, toward the center of the circle.
- This acceleration is called **centripetal** acceleration, $\boldsymbol{a_c}$
- $\boldsymbol{a_c}$ is opposite to $\boldsymbol{r}$ and ⊥ to $\boldsymbol{v}$


$\boldsymbol{v}$
$\boldsymbol{v}$
$\boldsymbol{a_c}$
$\boldsymbol{r}$

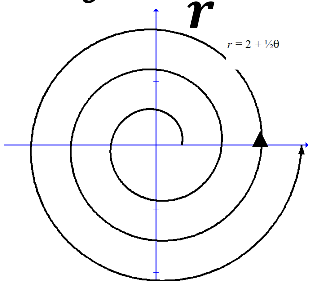
Uniform Circular Motion

- If the centripetal acceleration is too large, the object will spiral into the center.
- If it is too small, it will spiral outward.
- In between, there is a “Goldilock’s value” that is just right. At this value, the orbit closes on itself to form a circle. This value is:

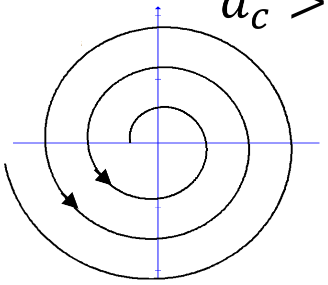
$$\boxed{a_c = \frac{\boldsymbol{v}^2}{\boldsymbol{r}}}$$

When motion is circular, an accel. of this size **must** be present

$a_c < \frac{\boldsymbol{v}^2}{\boldsymbol{r}}$

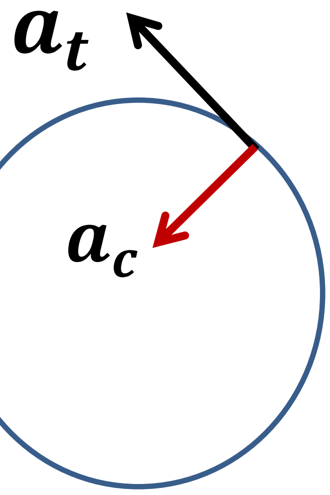


$a_c > \frac{\boldsymbol{v}^2}{\boldsymbol{r}}$



Uniform Circular Motion

- Uniform means the speed is constant. In that case **only** $\boldsymbol{a_c}$ is present. Remember that it changes the direction of $\boldsymbol{v}$, not the speed.
- If the circular motion is non-uniform (so speed is changing), there is also a **tangential** accel. $\boldsymbol{a_t}$, which changes the speed but not the direction.
- Our focus here is the centripetal acceleration. We'll come back to the tangential part in a later chapter.



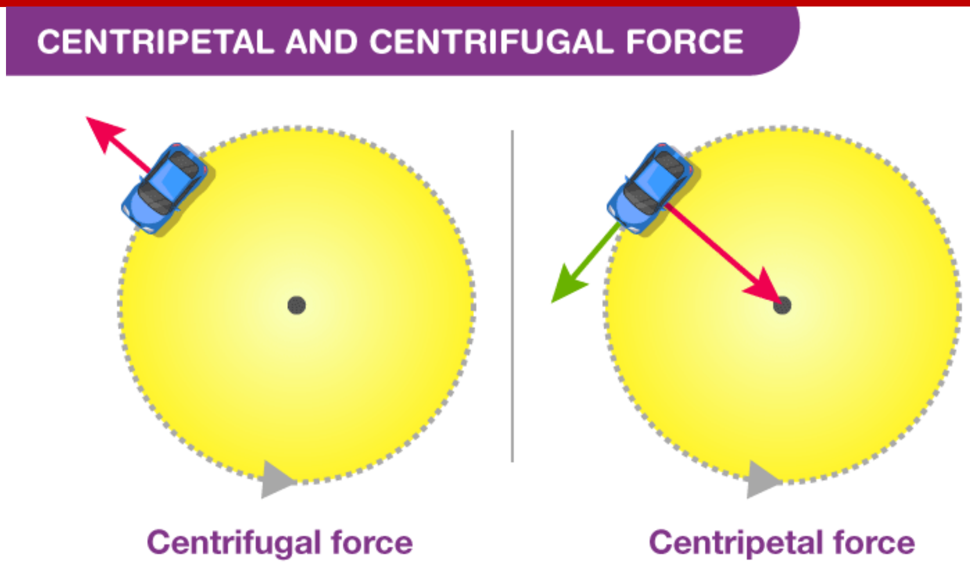
Uniform Circular Motion

- If there is a centripetal acceleration, then there must also be a centripetal force to cause it. This force points toward the center of the circle.
- This is not a new type force. It is one of the forces we are already familiar with (gravity, tension, normal force, spring force, etc)

Centripetal vs. Centrifugal

- Note that the force is **centripetal** (inward), not **centrifugal** (outward).
- Centrifugal forces are an example of a **fictitious force**: it is an artifact of viewing the situation from an accelerating coordinate system.
- In this case, the accelerating coordinate system is rotating with the observer (next slide)

Centripetal vs. Centrifugal


CENTRIPETAL AND CENTRIFUGAL FORCE
Centrifugal force
Centripetal force

- The wall and seat exert **inward** (centripetal forces) on passenger. Without them, passenger would travel in straight line (green arrow)
- From passenger’s perspective (rotating about center) he **appears** to be thrown **outward** into the door. In reality, the door is accelerating **inward** toward him.

Circular motion: more definitions

- **Period:**

  T = time to complete one trip around circle

  (units = seconds, hours, years, etc.)

- **Frequency:**

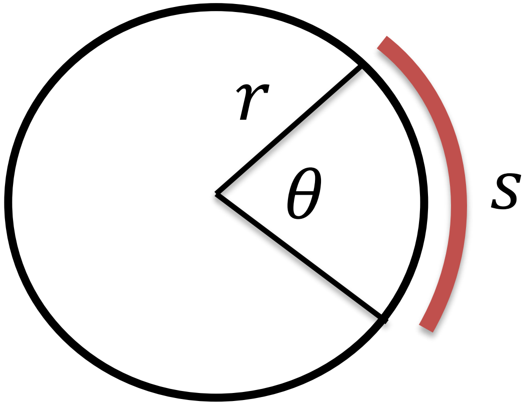
  f = number of trips around circle per unit time

  Most common unit: **Hertz** (1 Hz = $1\frac{rev}{s} = 1s^{-1}$)

- Basic relation: $\boxed{f = \frac{1}{T}}$

Distance versus angle

- **Arclength:** s = distance along portion of circle’s circumference (in m)
- **Angle:** θ (in radians)
- Relation between arclength and angle:



$$\boxed{s = r\theta}$$

Warning: This is **only** true θ if is in radians.

Example 1

- A rock of mass m=.1 kg is spun in a horizontal circle at the end of a string of length r = .5 m. The speed of the rock is constant: v=8 m/s

(a) Find the centripetal acceleration.

$$a_c = \frac{v^2}{r} = \frac{(8\frac{m}{s})^2}{.5\,m} = 128\ {}^{m}/{}_{s^2}$$

(b) What is the tension?

$$Tension = ma_c = (.1\,kg)\left(128\ {}^{m}/{}_{s^2}\right) = \boxed{12.8\,N}$$

(c) What are the frequency and period?

$$T = \frac{distance}{speed} = \frac{2\pi r}{v} = \frac{2\pi(.5\,m)}{8\,m/s} = \boxed{.392\,s}$$

$$f = \frac{1}{T} = \frac{1}{.392\,s} = \boxed{2.55\ \mathrm{Hz}}$$

Example 2

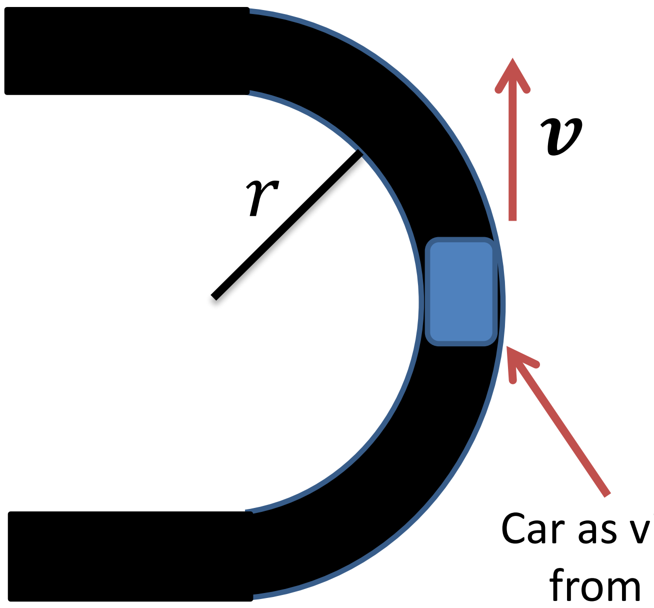
- What is the centripetal acceleration of the earth's orbit around the sun?
- Radius of orbit: $R = 1.5 \times 10^{11} m$
- Period of orbit: T=1 year= 3.16×10^{7} s

$$v = \frac{distance}{time} = \frac{2\pi R}{T} = \frac{2\pi(1.5 \times 10^{11} m)}{3.16 \times 10^{7} s} = 2.98 \times 10^{4} \frac{m}{s} = 29.8 \frac{km}{s}$$

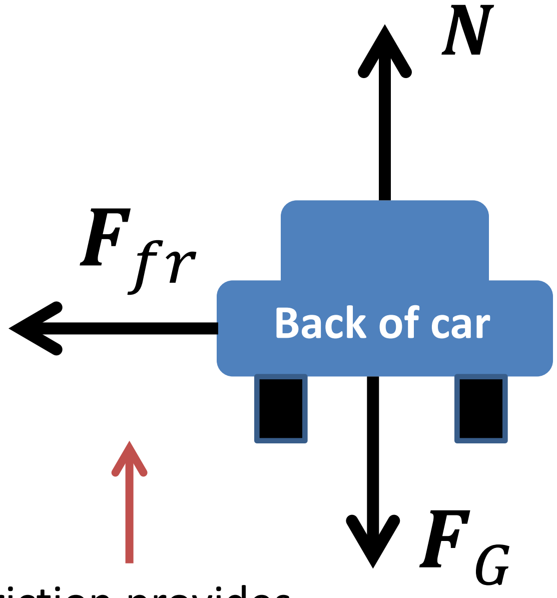
$$a_c = \frac{v^2}{R} = \frac{(2.98 \times 10^{4} \frac{m}{s})^2}{1.5 \times 10^{11} m} = \boxed{5.9 \times 10^{-3}\, {}^{m}/_{s^2}}$$

Example 3

- A car (mass m=1000 kg) travels around a circular bend in a horizontal road of radius $r = 80\ m$ at a speed of $v = 20\ {}^{m}/_{s}$. What is the minimum coeff. of static friction to keep the car from sliding off the road?


r
$\boldsymbol{v}$

Car as viewed from top


$\boldsymbol{N}$
$\boldsymbol{F}_{fr}$
Back of car
$\boldsymbol{F}_{G}$

Friction provides the centripetal force

Example 3 (continued)

$$\Sigma F_x = ma_x \quad \rightarrow \quad -F_{fr} = m(-a_c) \quad \rightarrow F_{fr} = ma_c$$

$$\Sigma F_y = ma_y \quad \rightarrow \quad N - F_G = 0 \quad \rightarrow N = F_G = mg$$

So:

$$\mu_S N \geq F_{fr} = ma_c = \frac{mv^2}{r}$$

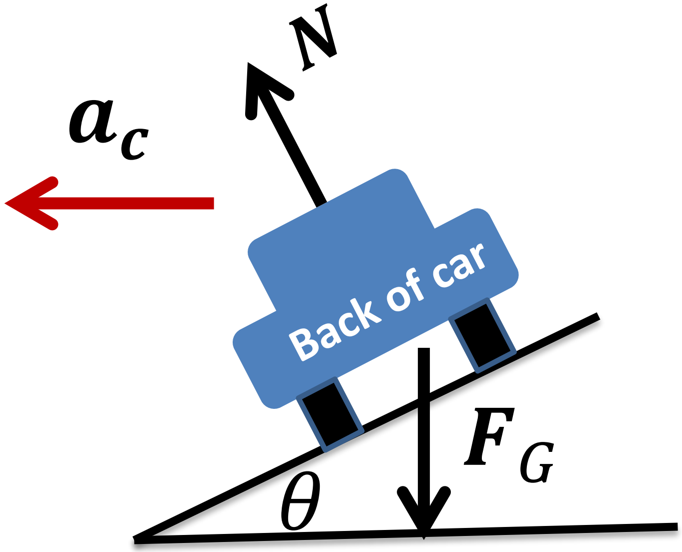
$$\mu_S \geq \frac{mv^2}{Nr} = \frac{mv^2}{mgr} = \frac{v^2}{gr} = \frac{(20)^2}{(9.8)(80)} = \boxed{.51}$$

Note that the required coefficient of friction is independent of mass. So light cars and heavy trucks can safely travel at the same speed on the same road.

Example 4

- Suppose the road of the previous example is now covered in ice, so $\mu_S \approx 0$.
- The car can still be kept on the road by banking (tilting) the road.
- By what minimum angle must it be banked?

- Recall: $r = 80m, v = 20\frac{m}{s}$
- Also: $a_x = -a_c,\ a_y = 0$
- This time, the centripetal force is provided by the horizontal part of the normal force



$\boldsymbol{N}$
$\boldsymbol{a_c}$
Back of car
$\boldsymbol{F}_G$
θ

Example 4 (continued)

$$\Sigma F_y = ma_y \quad \rightarrow \quad N\cos\theta - mg = 0 \quad \rightarrow \quad N = \frac{mg}{\cos\theta}$$

$$\Sigma F_x = ma_x \quad \rightarrow \quad -N\sin\theta = -m\frac{v^2}{r} \quad \rightarrow \quad N = \frac{mv^2}{r\sin\theta}$$

Set the two expressions for N equal, and solve for θ:

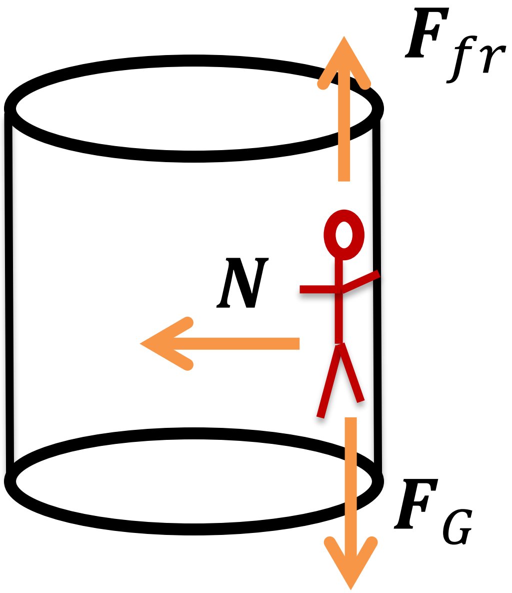
$$\frac{mg}{\cos\theta} = \frac{mv^2}{r\sin\theta} \quad \rightarrow \quad \frac{g\sin\theta}{\cos\theta} = \frac{v^2}{r} \quad \rightarrow \quad g\tan\theta = \frac{v^2}{r}$$

$$\theta = \tan^{-1}\frac{v^2}{gr} = \tan^{-1}\frac{(20)^2}{(9.8)(80)} = \tan^{-1}(.51) = \boxed{27°}$$

Again, the result is independent of the mass.

Example 5

- In an amusement park ride, a cylinder starts spinning, then the floor drops out. The people are against the wall and held up by friction. The radius of the drum is 2.5 m and the speed if the wall is 10 m/s. What is the minimum required to hold a person up?


$\boldsymbol{F}_{fr}$
$\boldsymbol{N}$
$\boldsymbol{F}_G$

The normal force plays the role of centripetal force

Set up F=ma equations and solve for friction coefficient

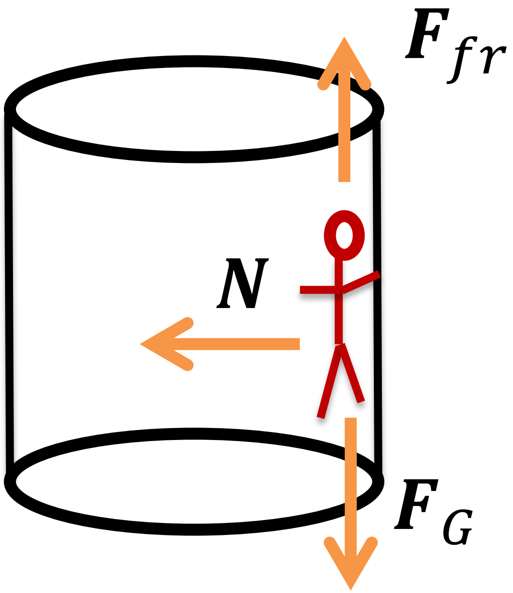
Example 5

$$\Sigma F_x = ma_x \rightarrow -N = -m\frac{v^2}{r} \rightarrow N = m\frac{v^2}{r}$$

$$\Sigma F_y = ma_y \rightarrow F_{fr} - mg = 0 \rightarrow F_{fr} = mg$$

- So:

$$\cancel{m}g = F_{fr} \leq \mu_S N = \mu_S \cancel{m}\frac{v^2}{r} \rightarrow g \leq \mu_S \frac{v^2}{r}$$

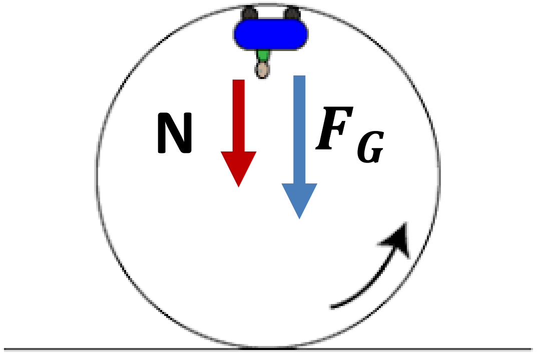


$$\mu_S \geq \frac{rg}{v^2} = \frac{(2.5)(9.8)}{10^2} = \boxed{.245}$$

This is again independent of the mass of the person

Example 6

A roller coaster goes around a 10m loop. At what speed must it go at the top of the loop for a passenger to remain in her seat without a seat belt?



$$\Sigma F_y = ma_y \quad \rightarrow \quad -N - mg = -m\frac{v^2}{r}$$

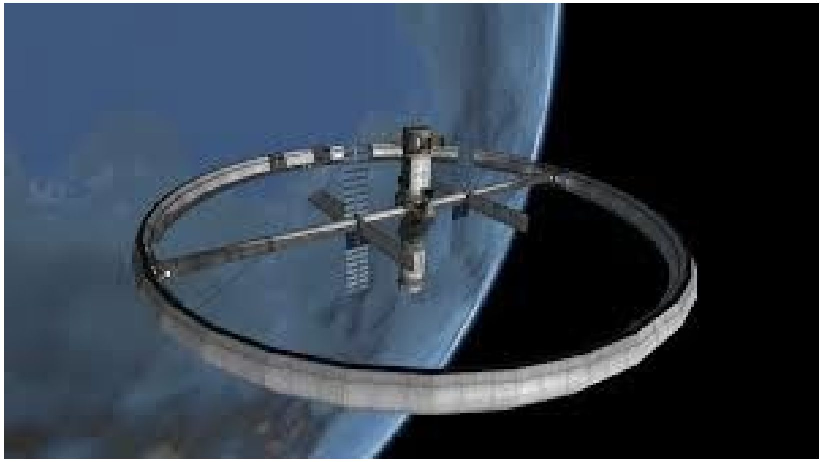
$$N + mg = m\frac{v^2}{r}$$

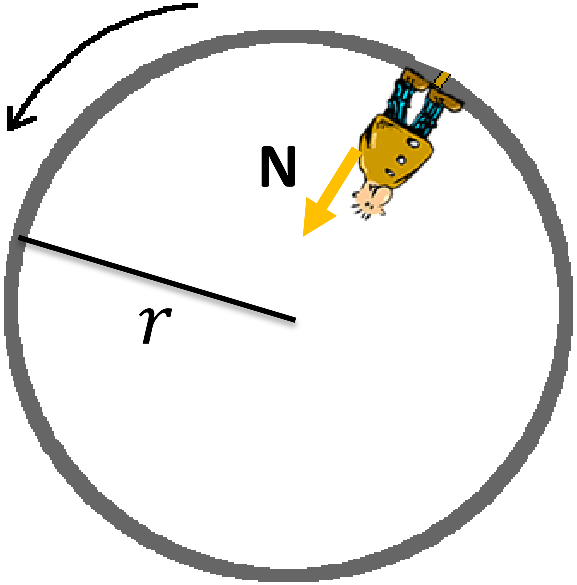
But $N \geq 0$, so: $mg \leq m\frac{v^2}{r} \quad \rightarrow g \leq \frac{v^2}{r}$

$$v \geq \sqrt{gr} = \sqrt{(9.8)(10)} = \boxed{9.9\ m/s}$$

Example 7

- A space station of radius r is rotating, in order to simulate gravity. The "gravity" is the normal force exerted by the floor.
- Want to have N = mg.





Example 7 (continued)

(a) At what frequency must station rotate to simulate Earth’s gravity?

$$\Sigma F = ma_c \quad \rightarrow \quad \mathrm{N} = \mathrm{m}\frac{v^2}{r} \quad \rightarrow \quad mg = \mathrm{m}\frac{v^2}{r} \rightarrow \quad g = \frac{v^2}{r}$$

But $v = \frac{2\pi r}{T} = 2\pi r f,$ so:

$$g = \frac{v^2}{r} = \frac{(2\pi r f)^2}{r} = 4\pi^2 r f^2$$

Solve for f:

$$f = \sqrt{\frac{g}{4\pi^2 r}} = \boxed{\frac{1}{2\pi}\sqrt{\frac{g}{r}}}$$

For example, if r = 1 km, then f=.016 Hz and T=63.5 s

Example 7 (continued)

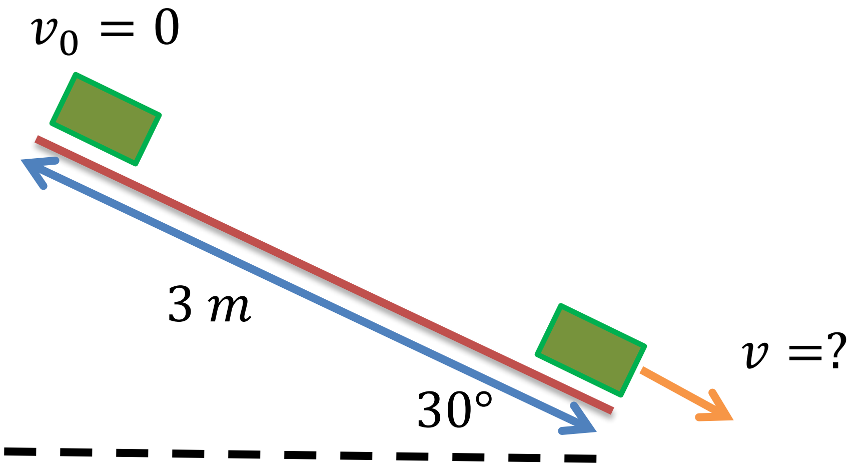
(b) Suppose r = 100m and f = .02 Hz. Find the normal force on a 50 kg astronaut.

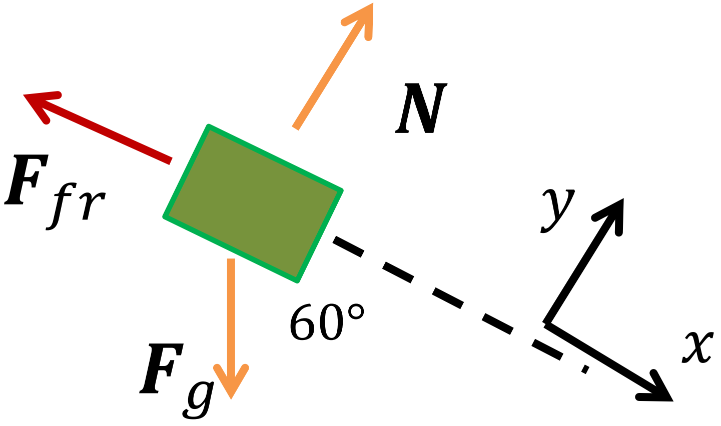
$$\mathrm{N} = \mathrm{m}\frac{v^2}{r} = m\frac{(2\pi rf)^2}{r} = 4\pi^2 mrf^2 = 78.9\ N$$

For comparison, the same person’s weight on Earth is $mg = (50)(9.8) = 490\ N$.

Extra friction example 1

- A block starts from rest at the top of a ramp inclined at 30° from the horizontal.
- Given: $\mu_K = .4\,, \quad m = 10\,kg$
- What is its speed when it reaches the bottom?




$\boldsymbol{N}$
$\boldsymbol{F}_{fr}$
$\boldsymbol{F}_g$
$60°$
y
x

Extra Friction Example 1

First find a, then find v.

$$\Sigma F_y = ma_y \quad \rightarrow \quad N - mg\sin 60° = 0$$

$$\rightarrow \quad N = mg\sin 60°$$

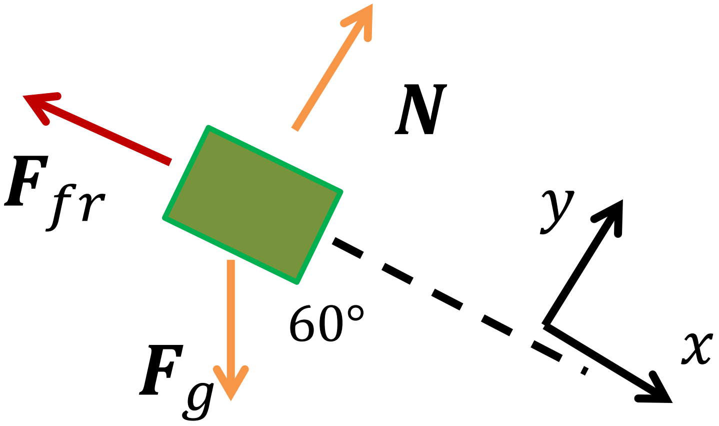
$$\Sigma F_x = ma_x \quad \rightarrow \quad mg\cos 60° - \mu_K N = ma$$

$$\rightarrow \quad mg\cos 60° - \mu_K mg\sin 60° = ma$$

$$\text{a} = g\cos 60° - \mu_K g\sin 60°$$

$$= (9.8)\cos 60° - (.4)(9.8)\sin 60°$$

$$= 1.5\ m/s^2$$

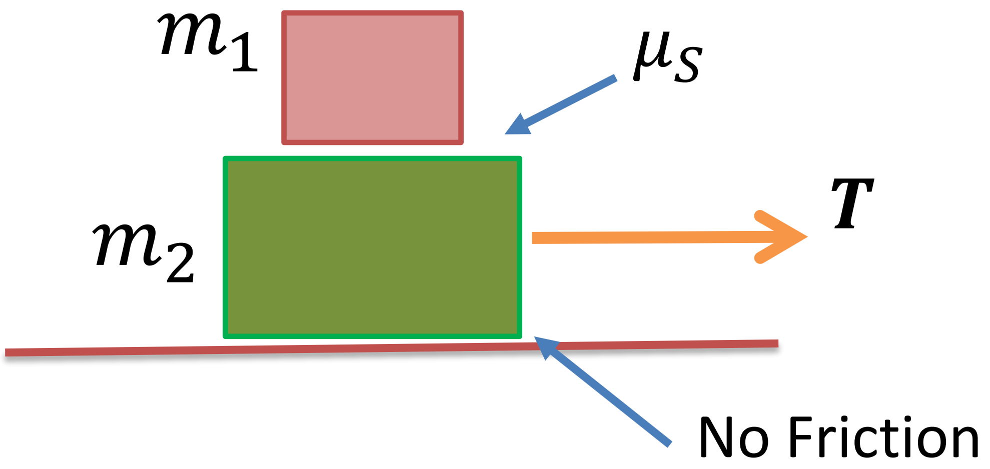


$\boldsymbol{N}$
$\boldsymbol{F}_{fr}$
y
x
60°
$\boldsymbol{F}_g$

$$v_x = \sqrt{v_{0x}^2 + 2a_x(x - x_0)}$$

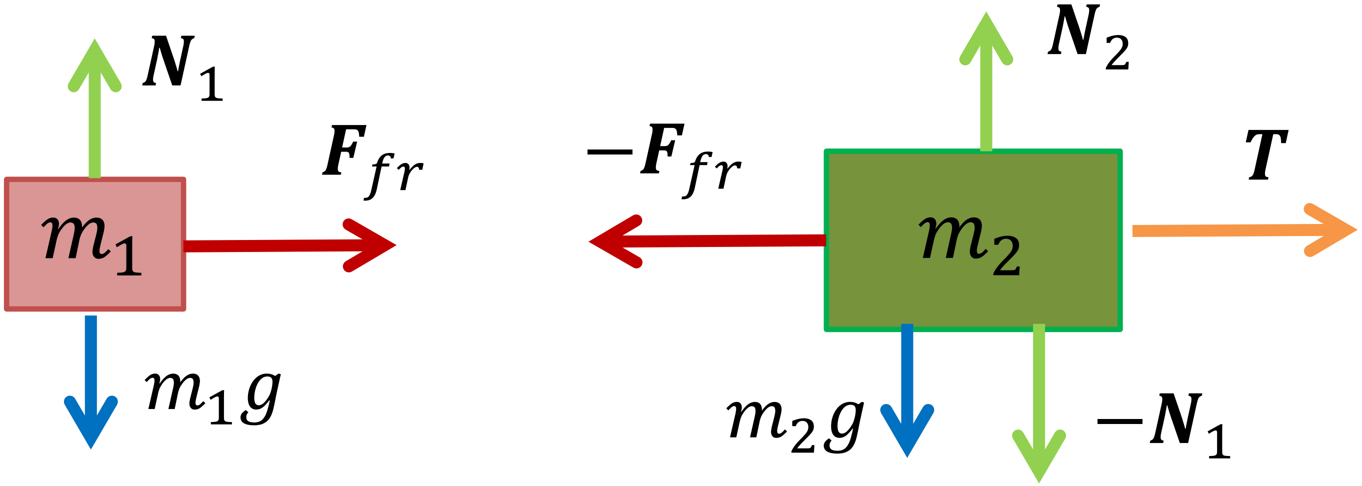
$$= \sqrt{0^2 + 2(1.5)(3)} = \boxed{3\ m/s}$$

Extra Friction Example 2


m_1
μ_S
m_2
$\boldsymbol{T}$
No Friction

- Small block on top of larger block. Coefficient of friction between them is known.
- Surface between bottom block and table is frictionless.
- Bottom block is pulled with a rope.
- What is the maximum tension for which the top block does not slide over the bottom one?

Extra Friction Example 2

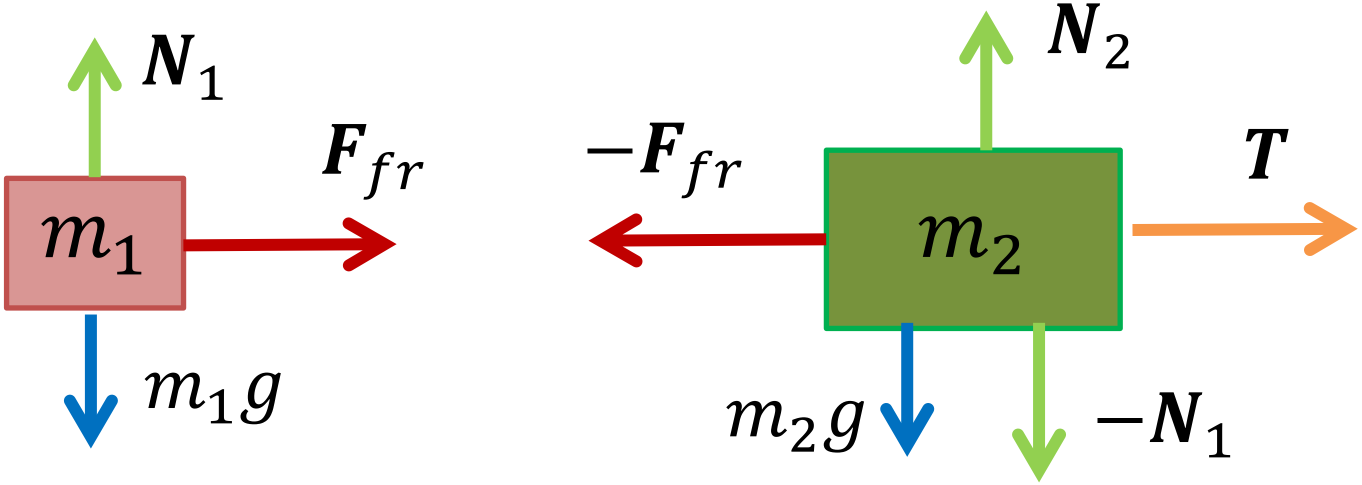

$\boldsymbol{N}_1$
$\boldsymbol{F}_{fr}$
m_1
$m_1 g$
$-\boldsymbol{F}_{fr}$
$\boldsymbol{N}_2$
m_2
$\boldsymbol{T}$
$m_2 g$
$-\boldsymbol{N}_1$

- No sliding between blocks → Both blocks have same **a**
- **a** only has an x-component (just call it a) . a_y vanishes.
- Top block: $N_1 = m_1 g$ Ⓐ

$$\mu_S N_1 = m_1 a \quad \rightarrow \quad \mu_S g = a$$ Ⓑ

Using Ⓐ

Extra Friction Example 2



$\boldsymbol{N}_1$
$\boldsymbol{F}_{fr}$
m_1
$m_1 g$
$\boldsymbol{N}_2$
$-\boldsymbol{F}_{fr}$
m_2
$\boldsymbol{T}$
$m_2 g$
$-\boldsymbol{N}_1$

- Bottom block:

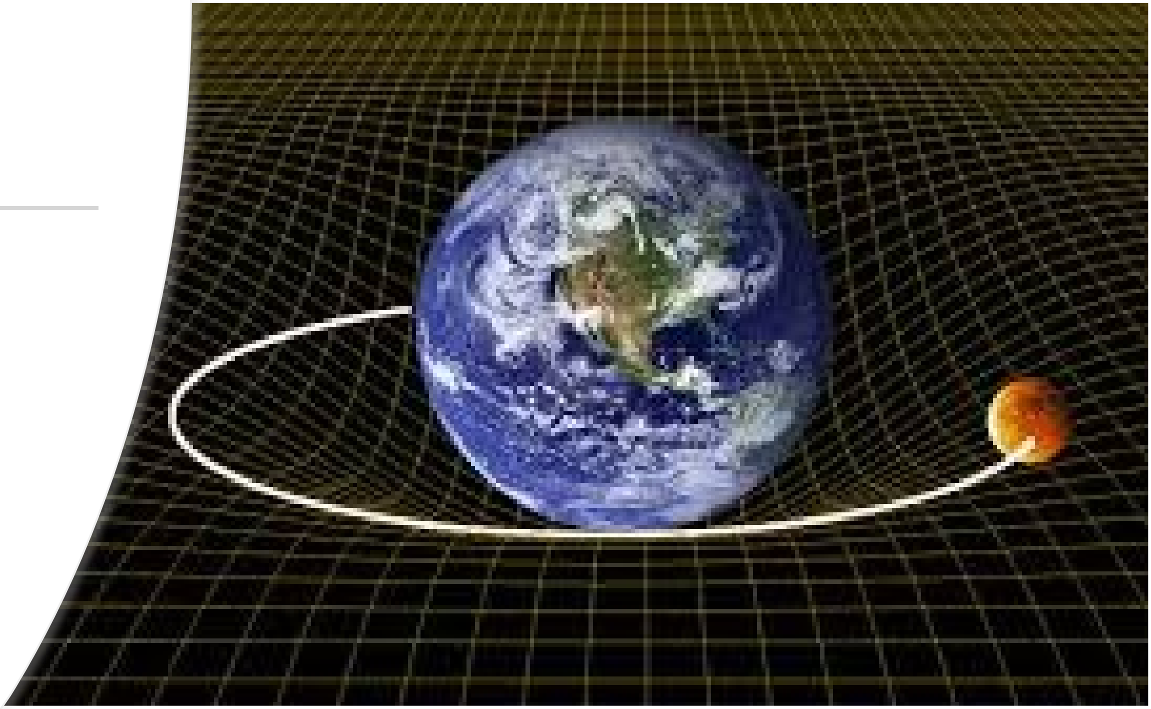
$$N_2 - N_1 - m_2 g = 0 \quad \rightarrow \quad N_2 = N_1 + m_2 g = (m_1 + m_2) g \quad Ⓒ$$

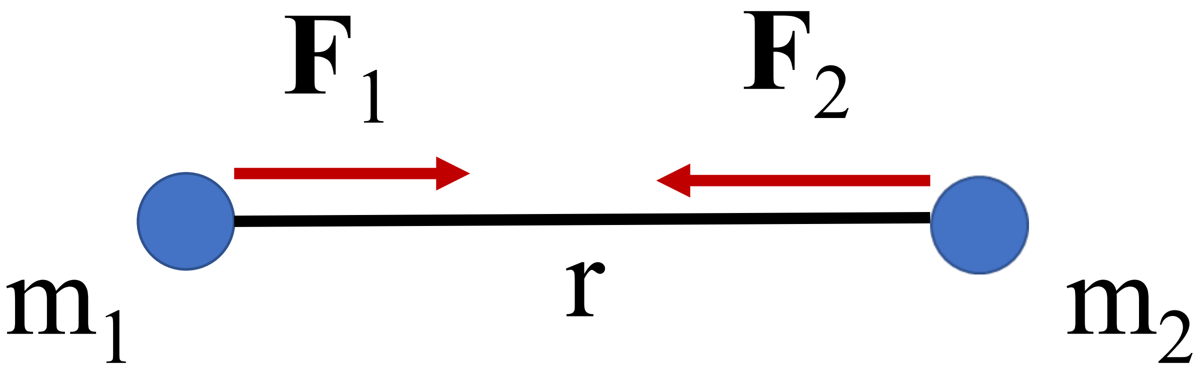
$$T - \mu_S N_1 = m_2 a \quad Ⓓ$$

- Use Ⓐ and Ⓑ in Ⓓ:

$$\boxed{T = \mu_S \, (m_1 + m_2) g}$$

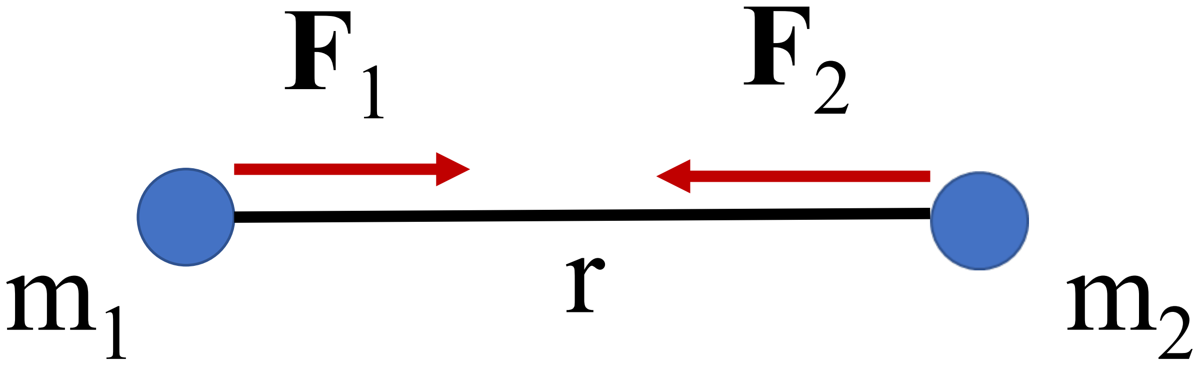
Ch. 6 – Newton’s Law of Gravity




$\mathbf{F}_1$
$\mathbf{F}_2$
m_1
r
m_2

- $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal and opposite forces.
- Magnitude: $F_G = G\,\frac{m_1 m_2}{r^2}$
- Newton's gravitational constant:

$$G=6.67\text{x}10^{-11}\ \text{Nm}^2/\text{kg}^2$$


$\mathbf{F}_1$
$\mathbf{F}_2$
m_1
r
m_2

In vector form:

$$\boldsymbol{F}_1 = G\,\frac{m_1 m_2}{r^2}\,\hat{\mathrm{r}}$$

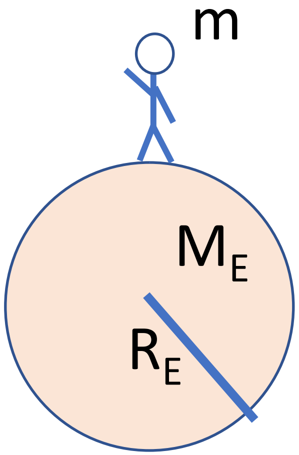
$$\boldsymbol{F}_2 = -G\,\frac{m_1 m_2}{r^2}\,\hat{\mathrm{r}},$$

where $\hat{\mathrm{r}}$ is unit vector point from m_1 to m_2 : $\hat{\mathrm{r}} = \mathbf{r}/\mathrm{r}$

Near surface of Earth

- Place object of mass m at surface of Earth.
- M_E and R_E are mass and radius of Earth.
- Then:

$$F_G = G\,\frac{m\,M_E}{R^2{}_E} = \mathrm{m}\left(G\,\frac{M_E}{R^2{}_E}\right) = \mathrm{mg}$$



- So:

$$\boxed{\mathrm{F}_G = \mathrm{mg}}$$

and

$$g = G\,\frac{M_E}{R_E^2} = \frac{(6.67\mathrm{x}10^{-11}\,\frac{N\,m^2}{kg^2})(5.97x10^{24}kg)}{(6.38x10^6\ m)^2} = \ 9.8\ \frac{m}{s^2}$$

$$\boxed{g = G\,\frac{M_E}{R_E^2} = \ 9.8\ \frac{m}{s^2}}$$

Examples:

1. An astronaut has m=75 kg. Find his weight on both the earth and the moon.

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Earth: $F_G = mg = (75\text{ kg})(9.8\text{ m/s}^2) = \boxed{735\text{ N}} = 165\text{ Lbs.}$

Examples:

1. An astronaut has m=75 kg. Find his weight on both the earth and the moon.

Earth: F_G = mg = (75 kg)(9.8 m/s^2) = $\boxed{735\text{ N}}$ = 165 Lbs.

Moon: M_m=7.35x10^{22} kg , R_m=1.74x10^6 m

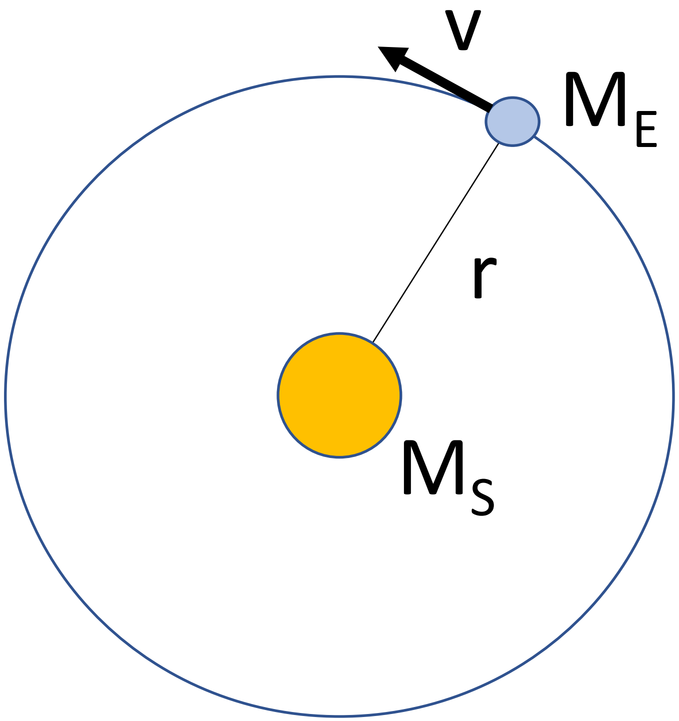
$$g_m = \frac{GM_m}{R_m^2} = \frac{(6.67x10{-}11)(7.35x1022)}{(1.74x106)^2} = 1.65\ \frac{m}{s^2}$$

$$F_G\ = mg_m = (75\ kg)\left(1.65\ \frac{m}{m^2}\right) = \boxed{121.44\text{ N}} = 27.3\text{ Lbs.}$$

Examples:

2. Given the following data:
   - Mass of Earth: $M_E = 5.97\text{x}10^{24}$ kg
   - Radius and period of its orbit: $r = 1.5\text{x}10^{11}$ m , $T = 1$ year $= 3.15\text{x}10^{7}$ s

- Approximate the Earth's orbit as a circle.
- Find: (i) the Sun's mass and (ii) the Earth's speed.



$M_S = ?$

$v = ?$

$$F_G = G\,\frac{M_S\,M_E}{r^2}$$

$$F_G = M_E\,a_c = M_E\,\frac{v^2}{r}$$

$$G\,\frac{M_S\,M_E}{r^2} = M_E\,\frac{v^2}{r}$$

$$G\,\frac{M_S}{r^2} = \frac{v^2}{r}$$ Ⓐ

Also: $v = \dfrac{2\pi r}{T}$ Ⓑ

From Ⓑ: $v = \dfrac{2(3.14)(1.5x1011\,m)}{3.15x107\,s} = \boxed{2.99x104\,\dfrac{m}{s}}$

From Ⓐ: $M_S = \dfrac{rv^2}{G} = \dfrac{(1.5x1011)(2.99x104)}{6.67x10^{-11}}$

$$\boxed{M_S\; = 2.01x10^{30}kg}$$

Examples:

3. Venus orbits the sun at an average radius of r=1.08x10^{11} m. How long is a year on Venus?

(Assume a circular orbit and use solar mass M$_S$=2.01x10^{30} kg.)

$$F = ma \quad \rightarrow \quad G\frac{M_V\, M_S}{r^2} = M_V\,\frac{v^2}{r} \quad \rightarrow \quad G\frac{M_S}{r^2} = \frac{v^2}{r}$$

So: $v = \sqrt{\frac{GM_S}{r}}$

Examples:

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So: $v = \sqrt{\frac{GM_S}{r}}$

But $2\pi r = vT$, which means:

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM_S}} = 2\pi\sqrt{\frac{(1.08x1011)^3}{(6.67x10^{-11})(2.01x1030)}}$$

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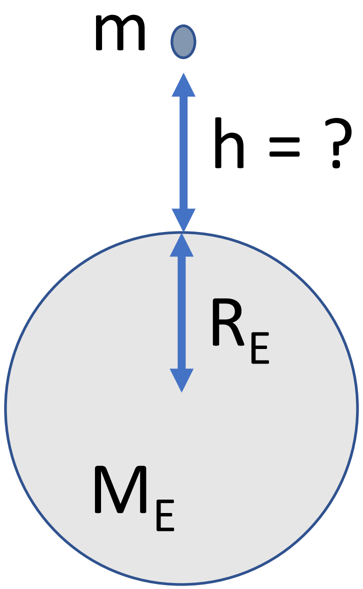
$$T = 1.93x10^7 s \; = \boxed{223\ days}$$

(Actual value: T=224.7 days)

Examples:

4. At what height above the earth's surface is the value of g equal to ½ its value at the surface?



At Earth's surface: $g = \frac{GM_E}{R_E^2}$

At $r = R_E + h$: $\frac{1}{2} g = \frac{GM_E}{(R_E+h)^2}$

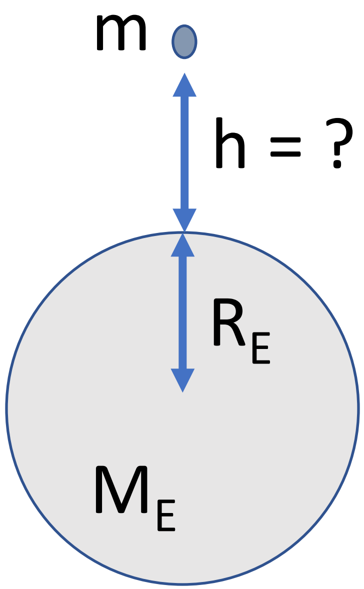
So: $\frac{1}{2}\frac{GM_E}{R_E^2} = \frac{GM_E}{(R_E+h)^2}$

$$\downarrow$$

$$2\,R_E^2 = (R_E + h)^2$$

Now solve the last equation for h

Examples:



$$2\,R_E^2 = (R_E + h)^2$$

$$\sqrt{2}\,R_E = R_E + h$$

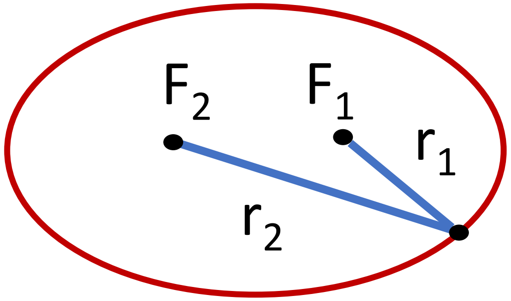
$$\mathrm{h} = \sqrt{2}\,R_E - R_E$$

$$\mathrm{h} = (\sqrt{2} - 1)\,R_E$$

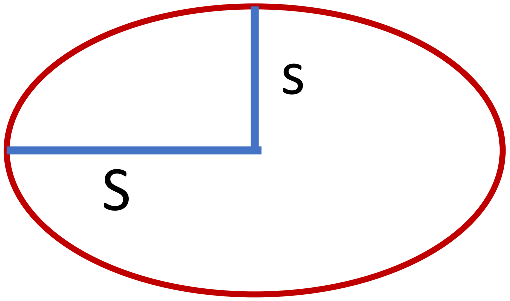
But R_E = 6370 km, so:

$$\boxed{\text{h= 2640 km}}$$

Elliptical orbits



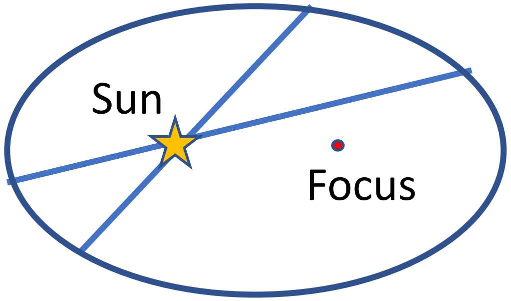
- Planetary orbits are ellipses.
- Recall properties of ellipses:
  - Two foci : F_1 and F_2
  - For any point on the ellipse, the sum of distances to the foci is constant: r_1+r_2 = constant
  - Semi-major and semi-minor axes:



  - As the two foci collide, the ellipse becomes a circle of radius r = s = S

Kepler's Laws

1. The path of each planet is an ellipse with the sun at one focus.
2. Each planet sweeps out equal areas in equal time.

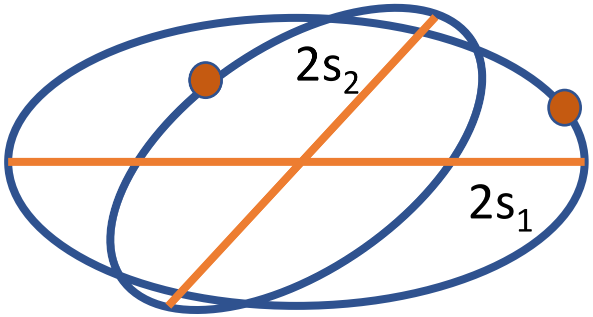


(Conservation of angular momentum)

3. Comparing the orbits of two planets:

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{s_1}{s_2}\right)^3,$$

where T = period, and s = semimajor axes



We essentially derived this in the last example for circular orbits:

$$T^2 = (\text{constant})\ r^3.$$

(The constant is $4\pi^2/GM_S$)

Apparent Weight

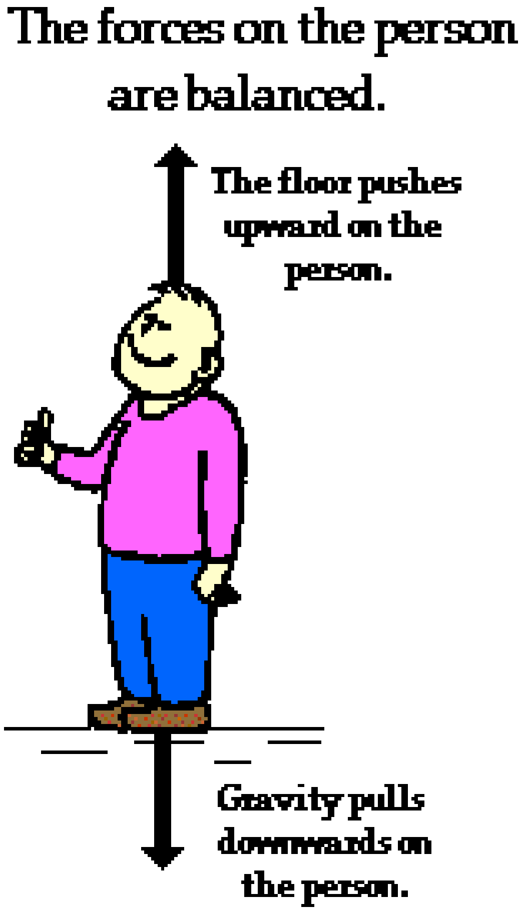
- What does a scale actually measure?
- Not weight, but normal force exerted between object and scale.
- But when the floor is level and the scale is at rest, normal force equals weight.

$$\Sigma \boldsymbol{F} = m\boldsymbol{a}$$

Take y-components:

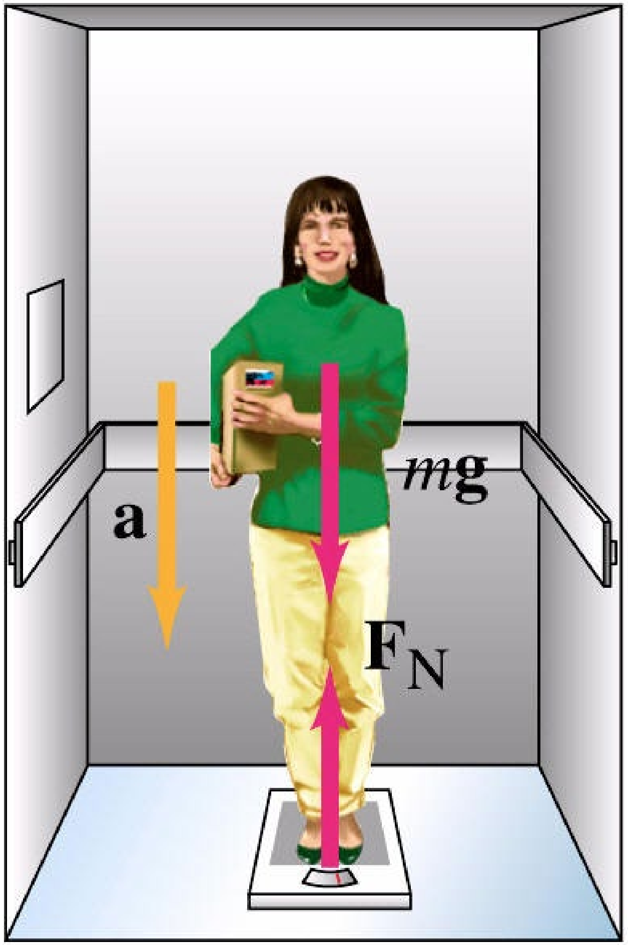
$$+N - mg = 0$$

So: $$\boxed{N = mg}$$


The forces on the person are balanced.
The floor pushes upward on the person.
Gravity pulls downwards on the person.

Apparent Weight

- But what if the scale is accelerating?
- Then the right side of ΣF=ma is not zero, which means N no longer equals the **actual** weight.
- How heavy we feel depends on how hard we are pushed against the floor (or seat, etc.)
- So the normal force measures our **apparent weight.**



Example:

Wile Coyote gets on elevator carrying his Acme ™ Brand Anvil.

The mass of Coyote + Anvil is: m= 60 kg.

a) For the first few seconds, the elevator accelerates upward with $a_y=+4\ m/s^2$. Apparent weight?

$$\boldsymbol{\Sigma F = ma} \quad \xrightarrow[\text{Y-components}]{} \quad N - mg = ma_y$$

So:

$$N = m(g + ay) = (60\ kg)(9.8 + 4)\ m/s^2 = \boxed{828\ N}$$

For upward acceleration, the apparent weight **is greater than** the actual weight.

Example:

b) Now the acceleration ends, and elevator continues at constant speed. What is the apparent weight, now?

Now, a=0, so using formula from part a:

$$N = m(g + ay) = (60\ kg)(9.8 + 0)\ m/s = \boxed{588\text{ N}}$$

In this case, the apparent weight **equals** the actual weight.

Example:

c) The coyote reaches the top of the building. As he prepares to drop the anvil on the passing roadrunner, he trips and falls. What is his new apparent weight on the way down?

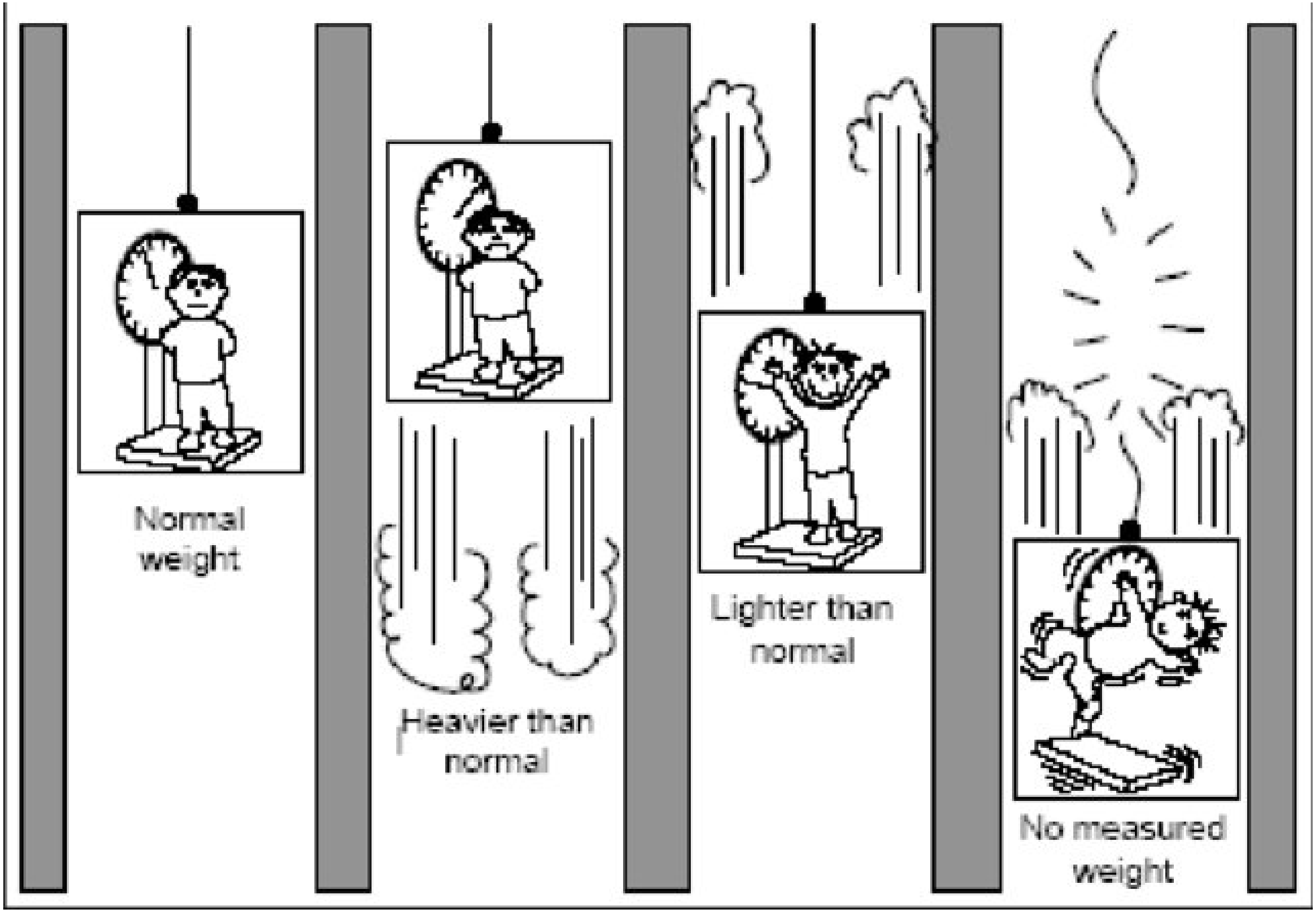
Now he is in free fall, so $a_y = -g$. So, as before:

$$N = m(g + ay) = (60\,kg)(g+(-g)) = \boxed{0\text{ N}}$$

When a is downward, apparent weight is **less than** actual weight.

When in free fall, object appears weightless!

Summary:

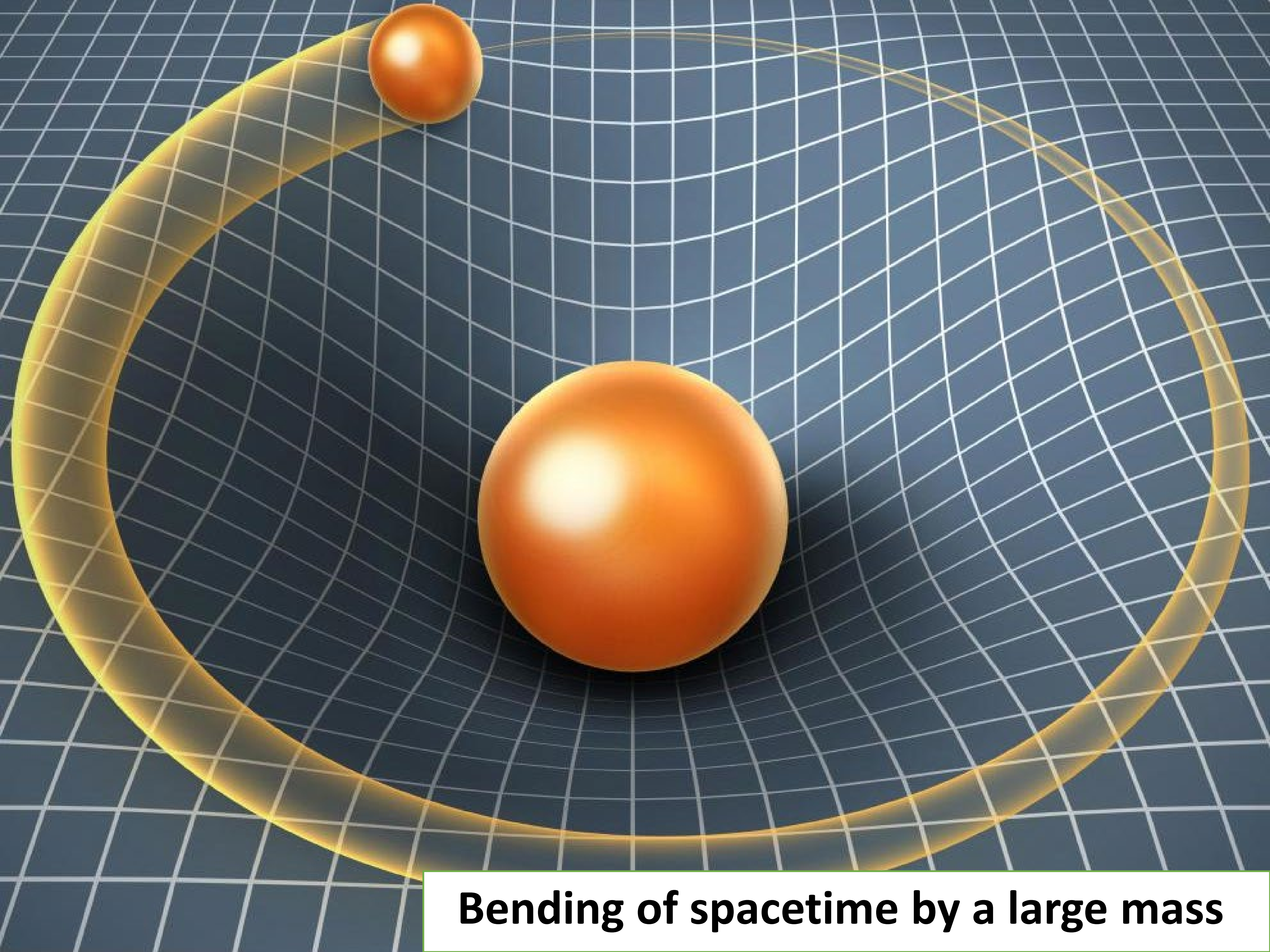

Normal weight
Heavier than normal
Lighter than normal
No measured weight

The Equivalence Principle

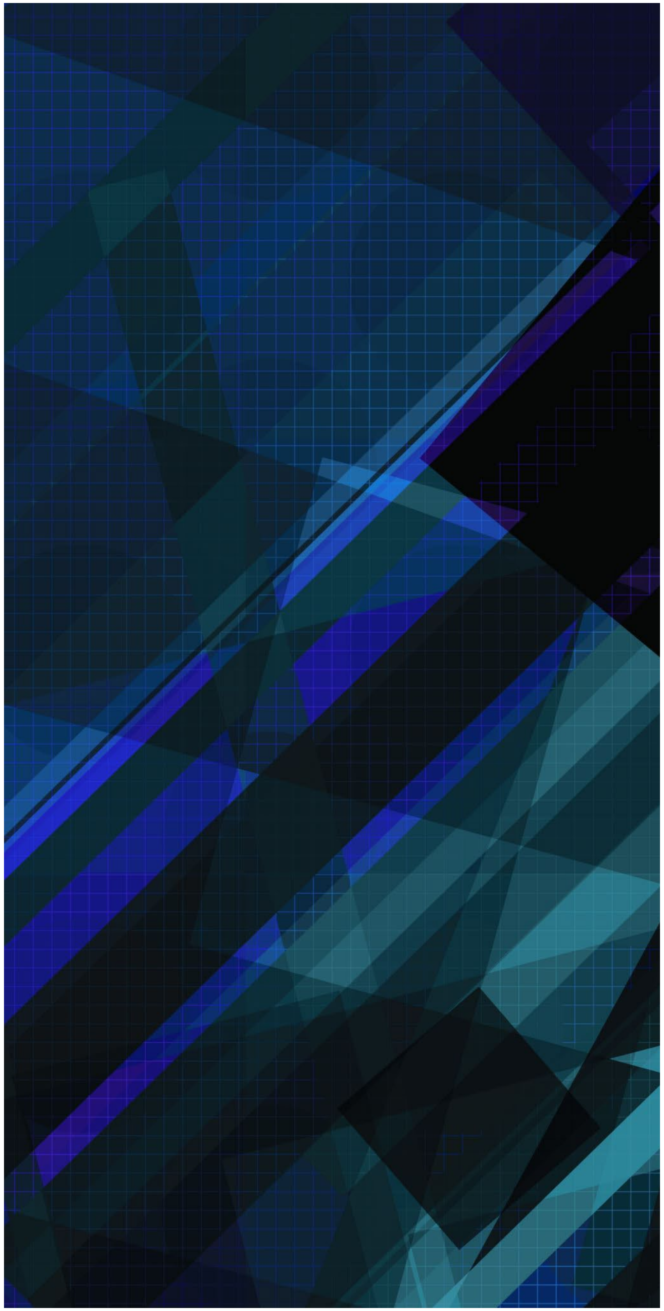
- In the last example, notice that gravity seemed to be canceled by acceleration when in free fall. This is an example of the **equivalence principle**:

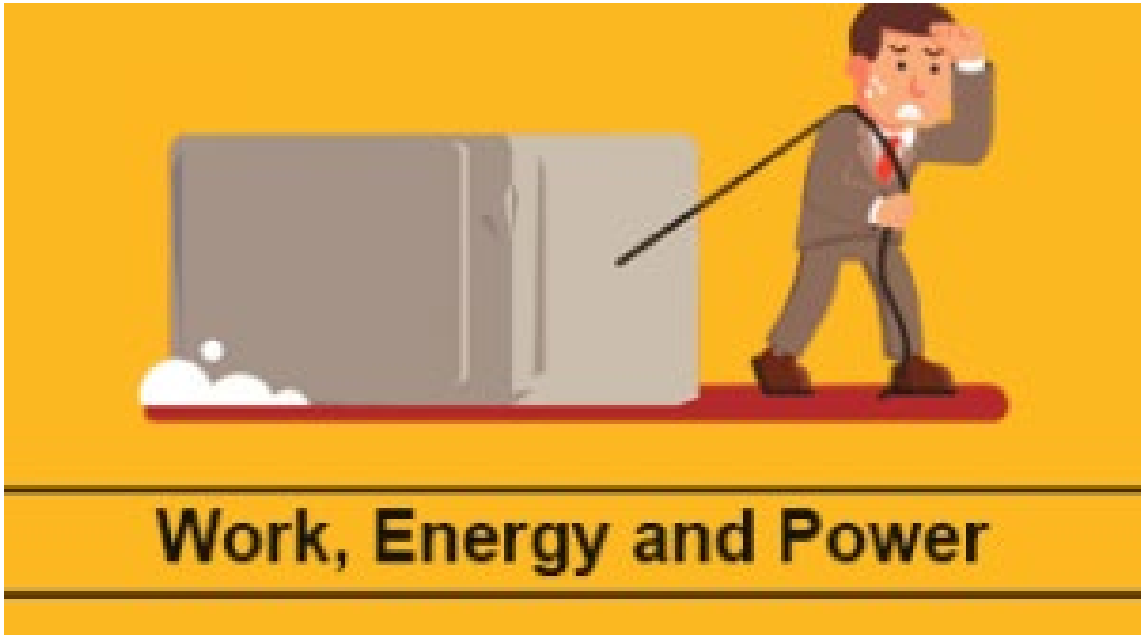
> Over small regions, it is impossible to distinguish between gravity and acceleration.

- This is the starting point for general relativity: what we see as gravity is actually an acceleration as a result matter following the bending of spacetime in the vicinity of nearby mass.
- In this view, gravity is a fictitious force, resulting from the rotation of our coordinate system as we follow the bending space.



Bending of spacetime by a large mass




Work, Energy and Power

Ch. 7 – Work and Kinetic Energy

- So far: Our main dynamical variables have been force and acceleration.
- These are both **vectors**.
  - **Advantage:** We get information about **direction** of motion.
  - **Disadvantage:** we need to keep track of components in each direction separately.
- Now we switch to dealing with **scalar** quantities.
  - **Advantage:** Simpler to work with: no components or angles.
  - **Disadvantage:** we lose information on direction of motion.
- Our main scalar quantities: **work** and **energy**.

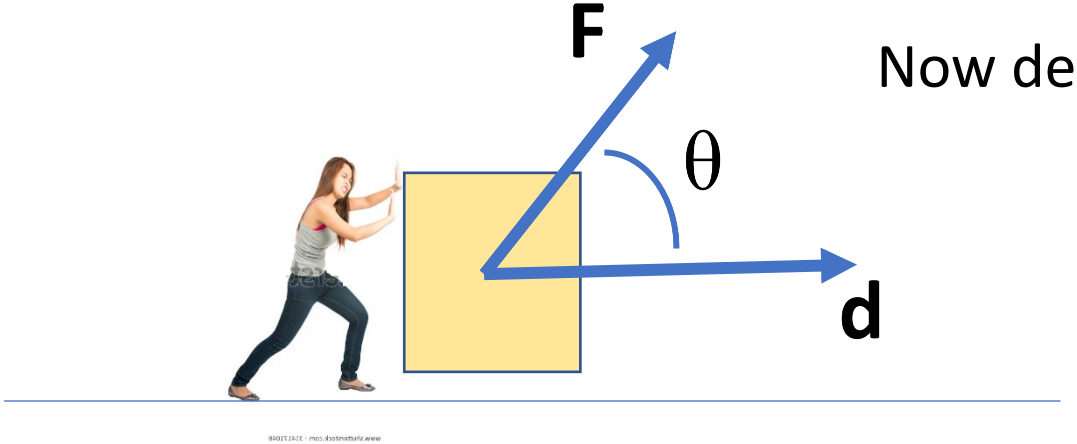
Units of work and energy

1. **Joules:** $1\,J \;=\; 1\,N\cdot m \;= 1\;\frac{kg\cdot m^2}{s^2}$
2. **Foot-pounds:** ft·lb
3. **Ergs:** 1 erg $=1\,\frac{g\cdot cm^2}{s^2}$

Conversions: 1 J = .738 ft lb = 10^7 ergs

Work

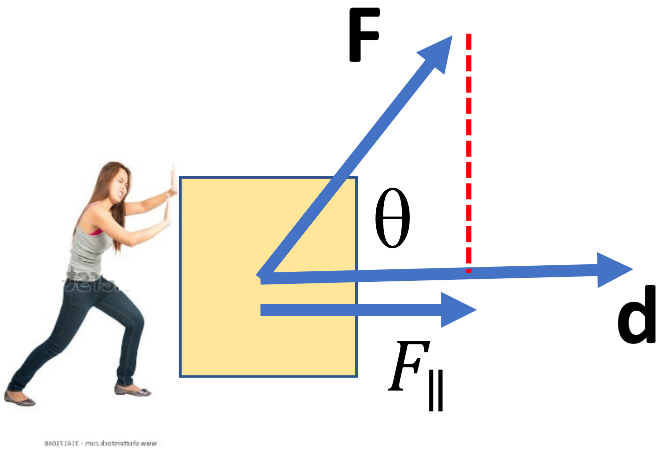
- An object moves along displacement vector: **d**
  - The magnitude of **d** is the distance moved.
  - The direction of d is the same as the direction of velocity.
- A force **F** is applied at angle θ from **d**.



Now define work

Work

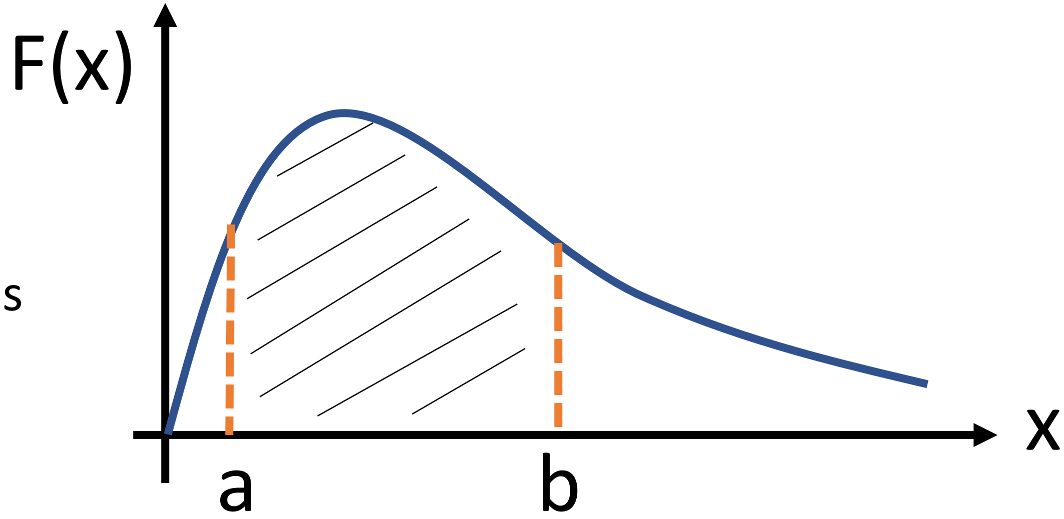
- General definition of work: $\boxed{W = \int \boldsymbol{F}(\boldsymbol{x}) \cdot \boldsymbol{dx}}$.
- If force is constant: $\boxed{W = \boldsymbol{F} \cdot \boldsymbol{d} = F_{\parallel} d,}$ where $F_{\parallel}$ is the component of **F** parallel to **d**: $F_{\parallel} = F\, cos\theta$



We will see below that the work measures energy changes as a result of a force.

Note that:

1. Work is zero if:
    1. F = 0 (no force)
    2. d = 0 (no motion)
    3. $F \perp d$ (no force in direction of motion)
2. If $\theta = 0$, then $W = \int F(x)dx$ is the area under the graph of F(x):

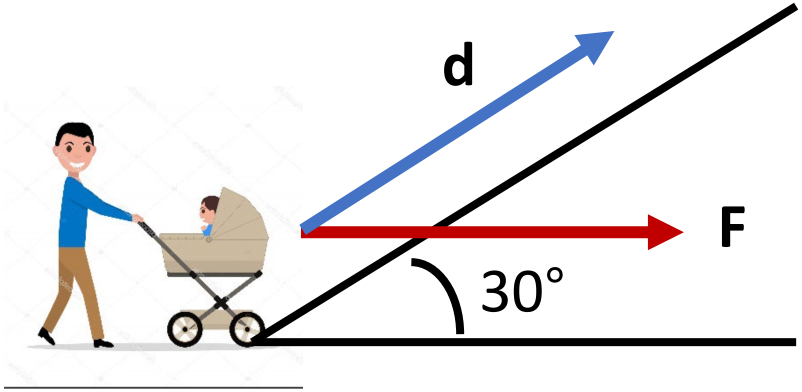


The shaded area gives the work done by the force as the object moves from x=a to x=b

Example

A parent pushes a 10 kg baby carriage up a 30° slope by exerting a horizontal force F=75 N.

a) How much work is done by F pushing the carriage 8 m up the hill?

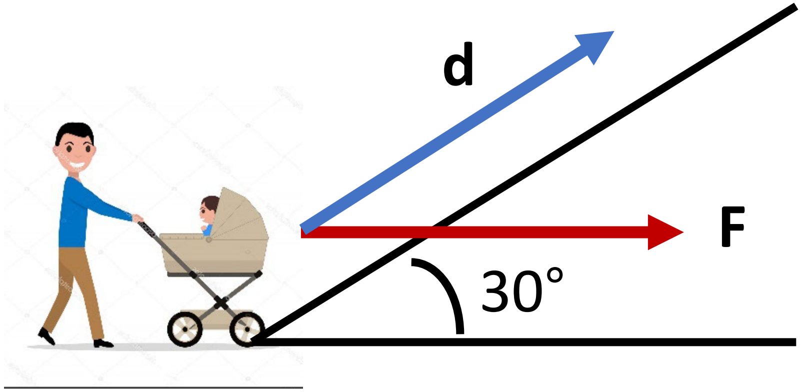


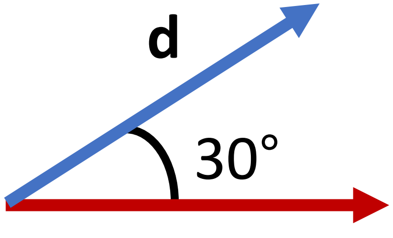
Example

A parent pushes a 10 kg baby carriage up a 30° slope by exerting a horizontal force F=75 N.

a) How much work is done pushing the carriage 8 m up the hill?

$$W = Fd\cos\theta = (75\,N)(8m)\cos 30^\circ = \boxed{519.62\,J}$$





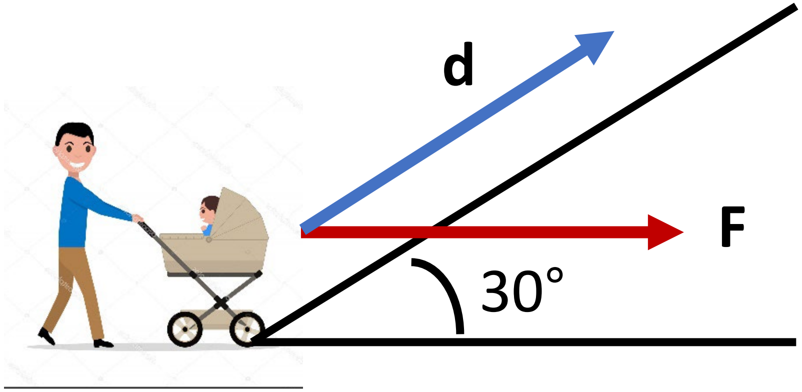
Example

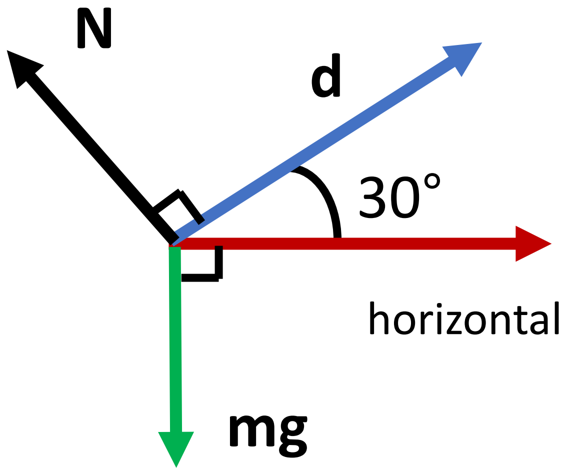
A parent pushes a 10 kg baby carriage up a 30° slope by exerting a horizontal force F=75 N.

b) How much work was done by gravity? By normal force?

$$W_G = mgd\cos\theta_G = (10\,kg)\left(9.8\,\frac{m}{s^2}\right)(8\,m)\cos 120° = \boxed{-392\,J}$$

$$W_N = Nd\cos\theta_N = N\,(8\,m)\cos 90° = \boxed{0\,J}$$





horizontal force F=75 N.

Sign of work

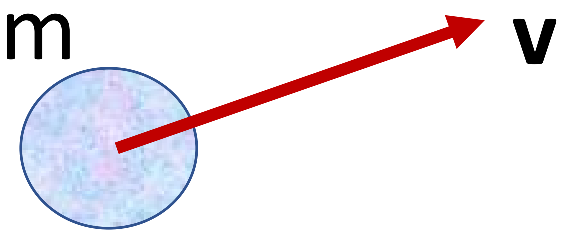
- The work can be positive or negative.
- W>0: The force adds energy to the system and aids the motion
- W<0: The force removes energy from the system and opposes the motion

- Note that we now have two views of forces:
  - Forces cause accelerations
  - Forces transfer energy between objects.

Kinetic Energy

- Kinetic energy is energy taken up my motion:

$$\boxed{K = \frac{1}{2}\mathrm{m}v^2}$$



- K has units of Joules
- Important: K is a **scalar**, not a vector. It has no direction or components.

- Work is a transfer of energy. More specifically:

$$\boxed{\textbf{Work-Energy Theorem:} \quad W = \Delta K}$$

($\Delta K = K_f - K_i$)

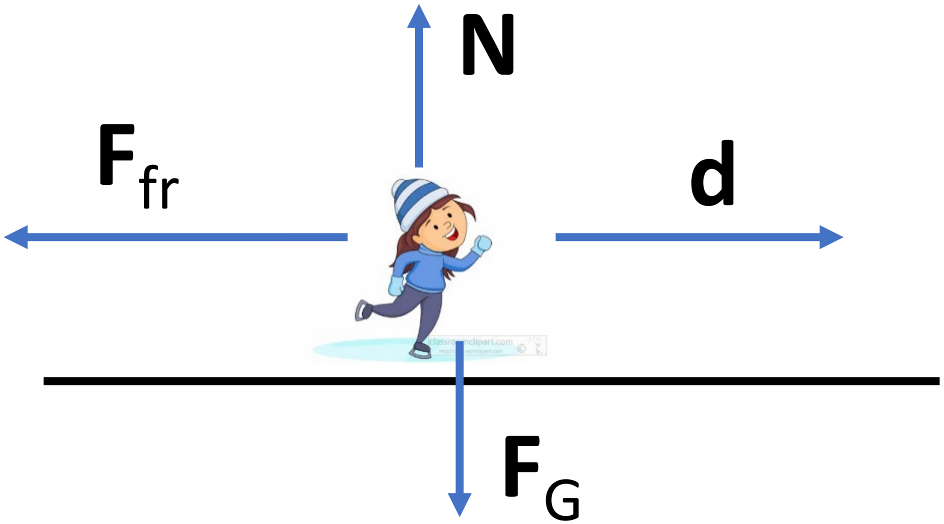
Example

A child of mass m=30 kg slides distance d=5 m across a patch of ice with μ_K=0.15 .

a) How much work is done by friction?

$$F_{fr} = \mu_K N = \mu_K mg$$

$$W = F_{fr} d\ cos180\circ$$
$$= \mu_K mgd\ cos180\circ$$
$$= (.15)(30\ kg)\left(9.8\ \frac{m}{s^2}\right)$$
$$* (5\ m)(-1)$$



W= - 220.5 J

Example

A child of mass m=30 kg slides distance d=5 m across a patch of ice with μ_K=0.15 .

b) If the child comes to comes to rest at the end, what was her speed?

$$K_f = \frac{1}{2}mv^2 = 0 \qquad K_i = \frac{1}{2}mv_0^2$$

$$W = \Delta K = K_f - K_i = 0 - \frac{1}{2}mv_0^2$$

$$v_0 = \sqrt{-\frac{2W}{m}} = \sqrt{-\left(\frac{2(-220.5\,J)}{30\,kg}\right)} = \boxed{3.83\ \frac{m}{s}}$$

Note that we can only find **speed**, not **velocity** by this method.

Example

A ball is thrown into the air (straight upward) with $v_0 = 10\ \frac{m}{s}$. How high does it go?

$$W = \boldsymbol{F}_G \cdot \boldsymbol{d} = mgd\cos 180^\circ = -mgd$$

$$W = \Delta K = K_f - K_i = 0 - \frac{1}{2}mv_0^2$$

At maximum height

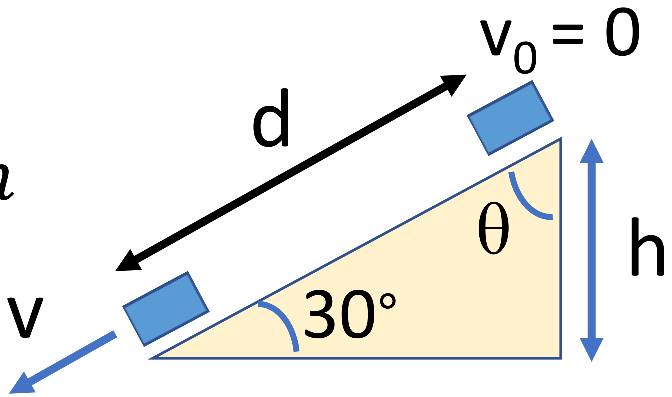
Set the two expressions for W equal to each other:

$$-\mathrm{mgd} = -\frac{1}{2}mv_0^2$$

$$d = \frac{v_0^2}{2g} = \frac{10^2}{2(9.8)} = \boxed{5.1\ m}$$

Example

A block slides down a ramp with no friction, starting from rest. The ramp is 5m high at the top. What is the speed when the block reaches the bottom?



$$\sin\theta = \frac{h}{d} \quad \rightarrow \quad d = \frac{5\ m}{\sin 30} = 10\ m$$

$$W = \boldsymbol{F}_G \cdot \boldsymbol{d} = mgd\cos 60^{\circ}$$

$$W = \Delta K \quad \rightarrow \quad mgd\cos 60^{\circ} = \frac{1}{2}mv^2 - 0$$

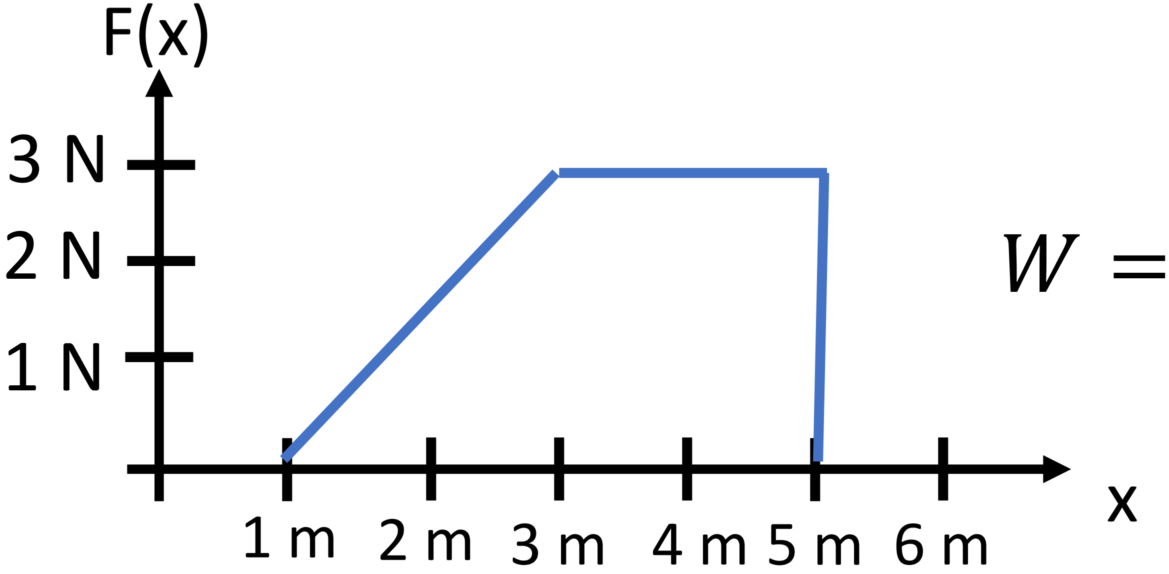
$$\mathrm{v} = \sqrt{2gd\cos 60°} = \sqrt{2(9.8\ )(10m)\cos 60°} = \boxed{13.5\ \frac{m}{s}}$$

Position-dependent forces

- Recall that when the forces depend on position, the more general definition of work is:

$$W = \int \boldsymbol{F}(x) \cdot \boldsymbol{dx}$$

- Suppose $\boldsymbol{F} \parallel \boldsymbol{dx}$. Then: $W = \int F(x) \cdot dx$ = area under of graph of F(x).



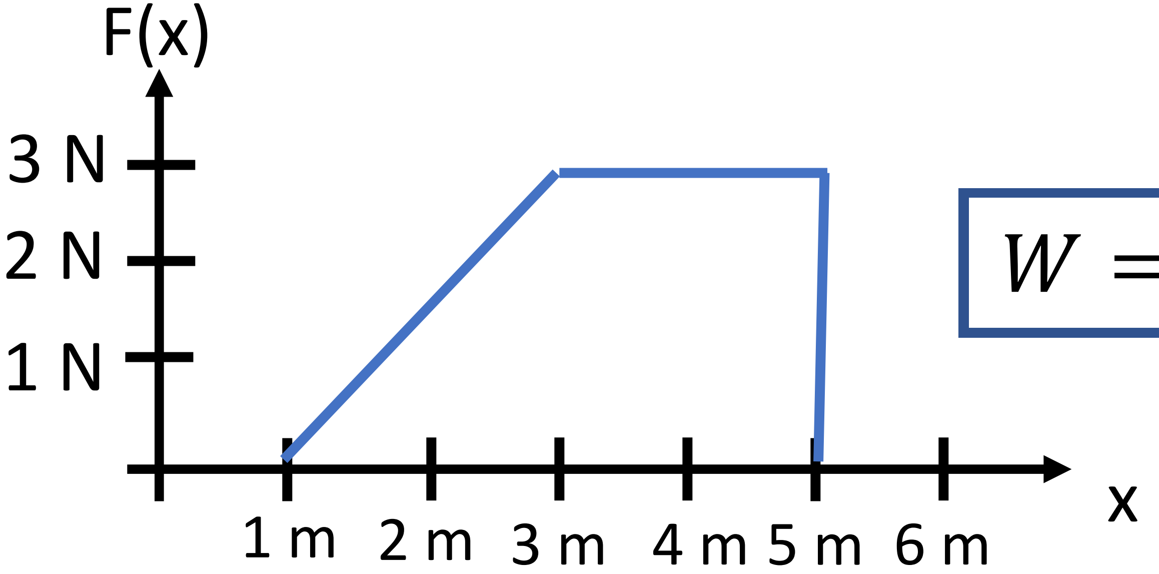
$W = ?$

Position-dependent forces

- Recall that when the forces depend on position, the more general definition of work is:

$$W = \int \boldsymbol{F}(x) \cdot \boldsymbol{dx}$$

- Suppose $\boldsymbol{F} \parallel \boldsymbol{dx}$. Then: $W = \int F(x) \cdot dx$ = area under of graph of F(x).



$$\boxed{W = 9\,J}$$

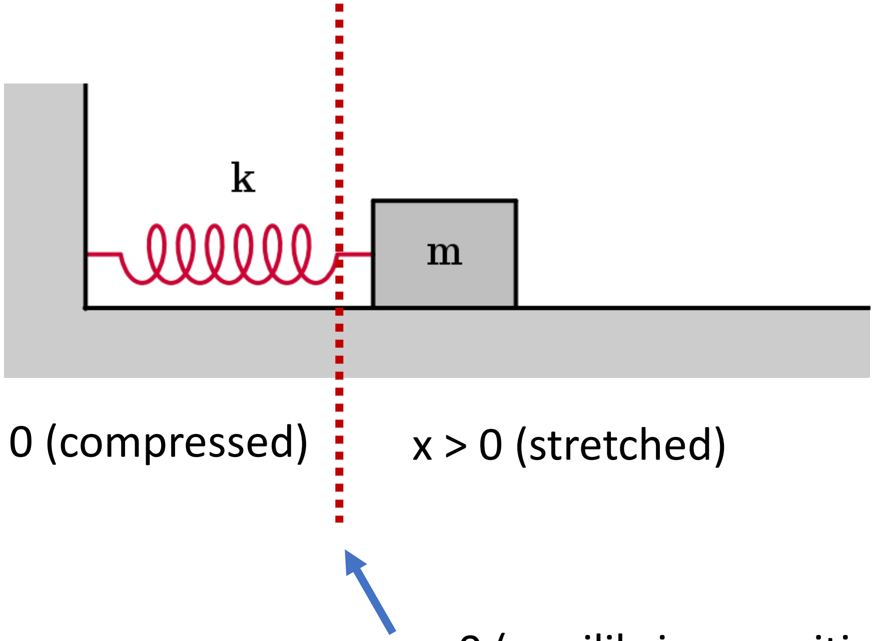
Now see an important special case of position-dependent force: ➡

Elastic forces

- Elastic forces (such as from stretched springs) obey **Hooke's law**:

$$\boxed{F = -kx}$$

Force is **opposite** to displacement



x < 0 (compressed)

x > 0 (stretched)

x =0 (equilibrium position)

- k = **spring constant** (or force constant) in N/m
- x = distance from equilibrium

Work done by elastic force

- Hooke's law: $\boldsymbol{F} = -k\boldsymbol{x}$
- Since F is a function of position, we **cannot** use $W = \boldsymbol{F} \cdot \boldsymbol{d}$.
- We **must** use the integral form: $W = \int \boldsymbol{F}(x) \cdot \boldsymbol{dx}$
- Work done by elastic force as mass moves from x_i to x_f :

$$\mathrm{W} = -\int_{x_i}^{x_f} k\,x\,dx \;=\; -\frac{1}{2}k\;x^2\Big|_{x_i}^{x_f}$$

$$\boxed{\mathrm{W} = \frac{1}{2}k\left(x_i^2 - x_f^2\right)}$$

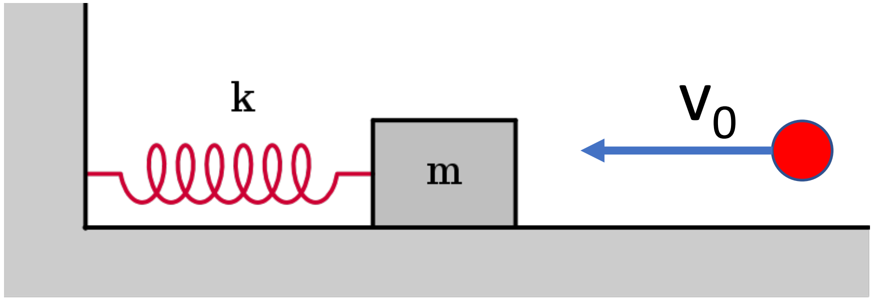
Example

- A mass moving at initial speed v_0 collides with an unstretched spring. How far will it be compressed?

$v_0 = 2$ m/s

$m = 1$ kg

$k = 16$ N/m



Give m initial speed vo-> How far does spring compress?

Momentarily at rest when maximum compression is reached

$$W = \Delta K \longrightarrow \frac{1}{2}k\left(\cancel{x_i^2} - x_f^2\right) = \frac{1}{2}m\left(\cancel{v_f^2} - v_0^2\right)$$

Spring is initially unstretched

$$\frac{1}{2}kx_f^2 = \frac{1}{2}mv_0^2 \longrightarrow x = \sqrt{\frac{m}{k}}v_0^2 = \sqrt{\frac{1}{16}}(2)^2 = \boxed{1\ m}$$

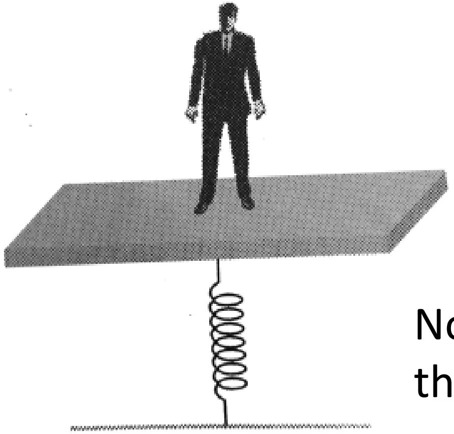
Example

A 70 kg man stands on a spring (k=3430 N/m), compressed by x= - 0.5 m . The spring is released.

a) What is his speed **immediately** after leaving the spring?

$$W = \Delta K \quad \rightarrow \quad \frac{1}{2}kx^2 = \frac{1}{2}m\, {v_f}^2$$

$$v_f = \sqrt{\frac{k}{m}}\, |x| = \sqrt{\frac{3430}{70}}\, |.5| = \sqrt{49}\,(.5) = \boxed{3.5\, m/s}$$



Note that here x (spring displacement) and y (height) are the same thing.

Example

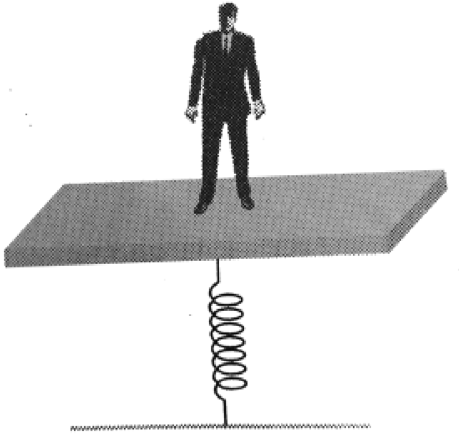
A 70 kg man stands on a spring (k=3430 N/m), compressed by x= - 0.5 m . The spring is released.

b) How high does he go? (Take initial height to be y=0.)

$$W = \Delta K \quad \rightarrow \quad -mgh = -\frac{1}{2} m\, {v_0}^2$$

$$h = \frac{v_0^2}{2g} = \frac{(3.5)^2}{2\,(9.8)} = \boxed{.625\, m}$$

(Final speed for part a) = (Initial speed for part b)



Power

- **Power** is the rate at which work is done, or equivalently, the rate at which something is changing in energy:

$$P = \frac{W}{\Delta t} = \frac{\Delta K}{\Delta t}$$

- Units are **Watts**: $1\,W = 1\,\frac{J}{s} = 1\,\frac{N\cdot m}{s}$

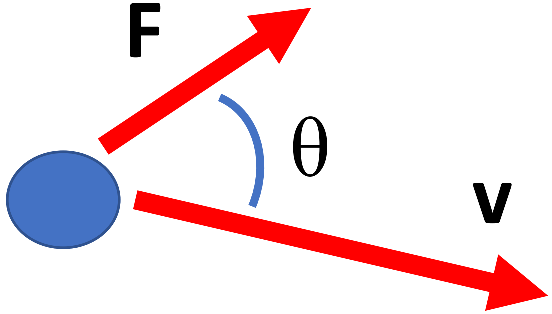
- Alternate units - **Horsepower**:

$$1\,Hp = 550\,\frac{ft\cdot lb}{s} = 746\,W$$

Power

- Since $W = F \, \triangle x \cos\theta$:

$$P = \frac{F \, \triangle x \cos\theta}{\triangle t} \longrightarrow \boxed{P = Fv\cos\theta}$$



- P gives the rate at which the force transfers energy to the moving object.

Example

- Suppose a single engine airplane has a 250 Hp motor.
  a) How much energy (in Joules) per hour is consumed by the engine from burning gasoline?

$$P = 250\ Hp\left(\frac{746\ W}{1\ Hp}\right) = 186{,}500\ W$$

$$P = \frac{\triangle E}{\triangle t} \quad \rightarrow \quad \triangle E = P \triangle t = (186{,}500W)(3600s) = \boxed{6.71x10^{8}J}$$

b) If the airplane's speed is $v = 200\ \frac{miles}{hr} = 90\ \frac{m}{s}$, how much force does the engine exert against air resistance?

- $P = Fv\cos 0 \quad \rightarrow \quad F = \frac{P}{v} = \frac{186{,}500\ W}{90\ m/s} = \boxed{2072.2\ N}$

Example

- A 2 kg block is sliding over a tale top. Its initial speed is $v_0 = 1\ \frac{m}{s}$, and it slides for time Δ t=.8 s before coming to rest.

a) Find the work done by friction and the power consumed.

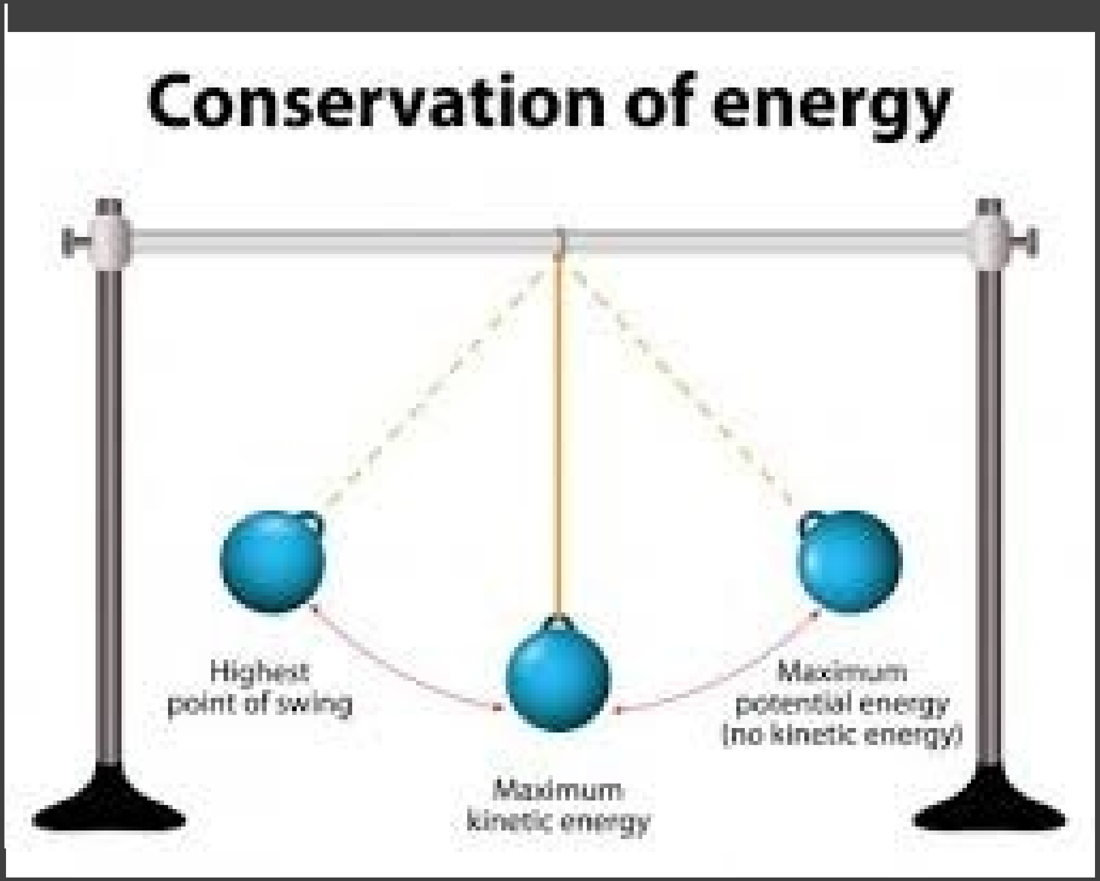
- $W = \triangle K = \frac{1}{2} m\, v^2 - \frac{1}{2} m\, {v_0}^2 = \frac{1}{2}(2kg)(0^2 - 1^2) = \boxed{-1\ J}$

$$P = \frac{\triangle K}{\triangle t} = \frac{W}{\triangle t} = \frac{-1\ J}{.8\ s} = \boxed{-1.25\ W}$$

b) If the total distance the block slides is d = 2m, find the friction force and the coefficient of friction.

- $W = -F_{fr} d \quad \rightarrow \quad F_{fr} = -\frac{-1\ J}{2\ m} = 0.5\ N$
- $F_{fr} = \mu_K\, F_N = \mu_K\, mg \quad \rightarrow \quad \mu_K = \frac{F_{fr}}{mg} = \frac{.5\ N}{(2kg)(9.8\frac{m}{s^2})} = \boxed{.025}$

Ch. 8 – Potential Energy & Energy Conservation


Conservation of energy
Highest point of swing
Maximum potential energy (no kinetic energy)
Maximum kinetic energy

ENERGY

- **Energy** = "ability to do work"
- Energy is a **scalar**, measured in Joules
- Can split energy into two main types:

Energy = (Mechanical energy) +
(Nonmechanical energy)

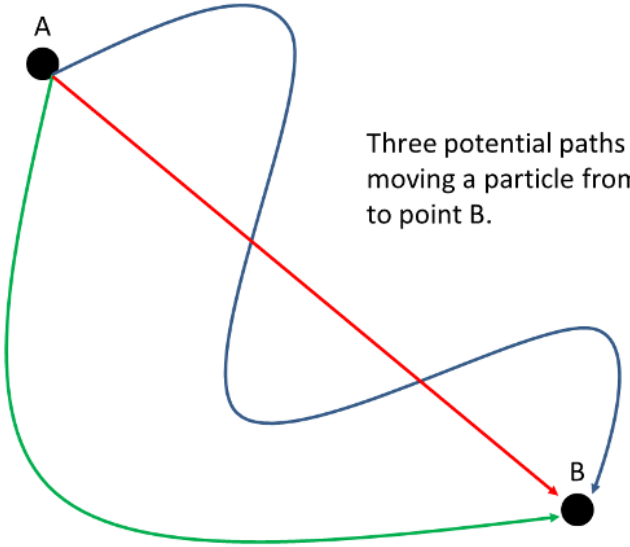
- Examples of nonmechanical energy:
  - Heat, Light, Sound, Nuclear
- We will come back to heat later in the semester
- For now, focus on mechanical energy

Mechanical energy

- Mechanical energy, E: $E = K + U$
  - Kinetic energy: $K = \frac{1}{2}mv^2$
  - Potential energy: U = stored energy (associated with a *conservative* force)
- There are many types of potential energies, associated with different forces. The form of U depends on the specific force.

Conservative Forces

- **Conservative force:** Work is dependent only on the initial and final points, not on the path between them.
- Conservative examples:
  - Gravity
  - Electric and magnetic forces
  - Elastic forces
- Nonconservative examples:
  - Friction, viscosity, air resistance



Three potential paths for a force moving a particle from point A to point B.

Potential energy and conservative forces

- ***Only*** conservative forces have associated potential energies.
- We'll see below that conservative forces conserve mechanical energy: $\Delta E = 0$
- Nonconservative forces convert mechanical energy into nonmechanical energy.
  - Prime example: **Friction** converts kinetic energy (mechanical) into heat (nonmechanical)

Finding Potential Energy

- Given a **conservative** force, the general form of potential energy is:

$$\Delta U = -\int_A^B \boldsymbol{F}(x) \cdot \boldsymbol{dx}$$

- Note that only **changes** in U are physically relevant.
- To define U (as opposed to ΔU), must choose a reference point at which U is **defined** to be zero.
- Physical quantities do not depend on this choice of reference point.

Gravitational Potential Energy (near Earth's surface)

- $\boldsymbol{F} = -mg\hat{\boldsymbol{j}}$
- $\Delta U = -\int_{\boldsymbol{y_0}}^{\boldsymbol{y}} \boldsymbol{F} \cdot \boldsymbol{dy} = +\boldsymbol{mg}(\boldsymbol{y} - \boldsymbol{y_0}) = \boldsymbol{mg}\Delta\boldsymbol{y}$
- Reference point: Choose U=0 at y_0=0, so:

$$U(y) - \cancel{U(0)} = mg(y - \cancel{y_0})$$

$$\boxed{U(y) = mgy}$$

Elastic Potential Energy

- $F = -kx$
- $\Delta U = -\int_{x_0}^{x} F\, dx = +\int_{x_0}^{x} kx\, dx = \frac{1}{2}k(x^2 - x_0^2)$
- Reference point: U = 0 when x_0 = 0. So:

$$U(x) - \cancel{U(x_0)} = \frac{1}{2}k(x^2 - \cancel{x_0^2})$$

$$\boxed{U(x) = \frac{1}{2}kx^2}$$

Relation of U to K and W

- For a conservative force: $W = \Delta K = -\Delta U$
- Note therefore that:

$$\Delta E = \Delta K + \Delta U = (+W) + (-W) = 0$$

Conservative forces conserve mechanical energy:

$$\Delta E = \Delta K + \Delta U = 0$$

- Nonconservative forces (friction) can change E, since there is no ΔU to cancel the ΔK:

$$\Delta E = \Delta K \neq 0$$ (Nonconservative force)

Conservation of energy

- **Conservative forces** conserve mechanical energy
- **Nonconservative forces** convert between mechanical and nonmechanical energies
- **Total energy** (mechanical +nonmechanical) is **always** conserved.

ENERGY

- Mechanical
  - Kinetic
    - Linear
    - Rotational
  - Potential
    - Gravitational
    - Elastic
    - Electric
    - Magnetic
- Non- Mechanical
  - Light
  - Sound
  - Heat

- **Example**

What is the initial speed of a rock thrown vertically upward if it reaches a maximum height of 4 m above its starting point?

$$0 = \Delta K + \Delta U = (K - K_0) + (U - U_0)$$

$$K_0 + U_0 = K + U$$

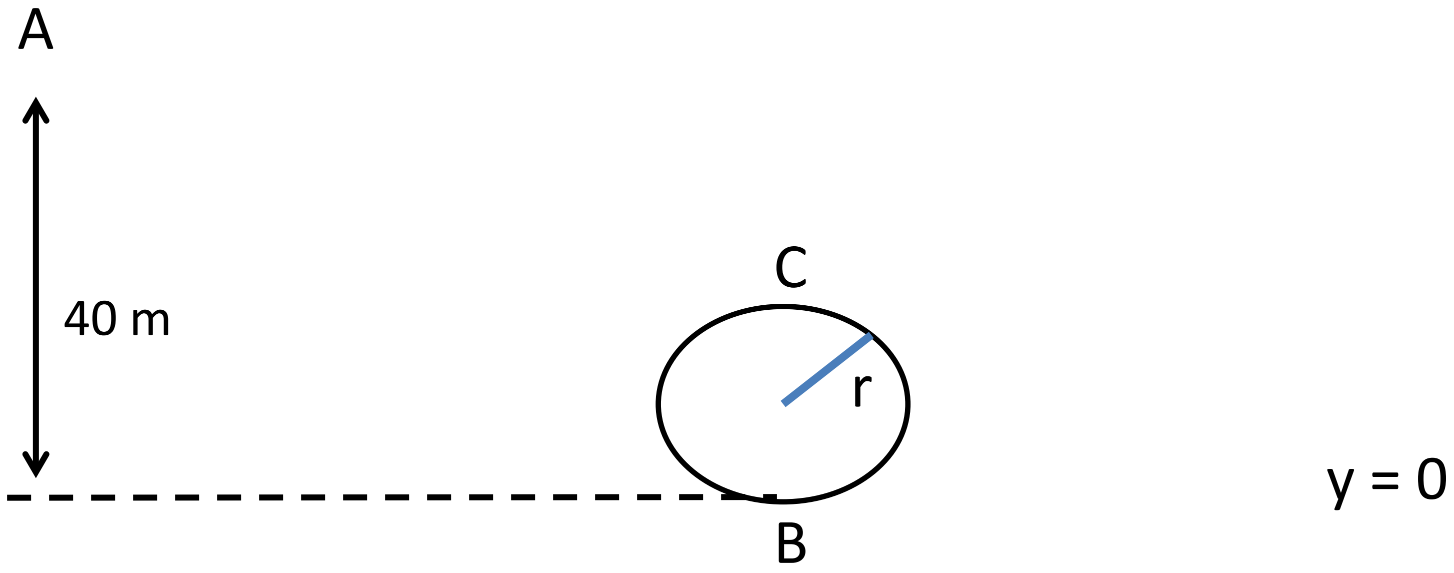
v=0 at maximum height

$$\frac{1}{2}mv_0^2 + \cancel{mgy_0} = \cancel{\frac{1}{2}mv^2} + mgy$$

$$v_0 = \sqrt{2gy} = \sqrt{2(9.8)(4)} = \boxed{8.85\ m/s}$$

- **Example**

A roller coaster starts at rest at point A. The track is frictionless, and the circle has a radius of 10 m. Find the speed at points A and B.



$$E_A = K_A + U_A = 0 + mgy_A \qquad E_B = \frac{1}{2}mv_B^2 + 0$$

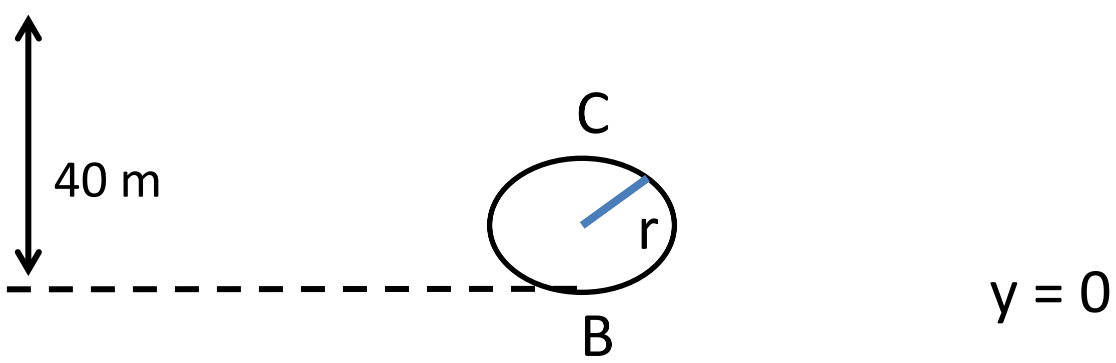
$$E_C = \frac{1}{2}mv_C^2 + \mathrm{mg}y_C = \frac{1}{2}mv_C^2 + 2\mathrm{mgr}$$

Point B: $E_A = E_B \rightarrow mgy_A = \frac{1}{2}mv_B^2$

$$v_B = \sqrt{2gy_A} = \sqrt{2(9.8)(40)} = \boxed{28\,m/s}$$

Point C: $E_A = E_C \rightarrow mgy_A = \frac{1}{2}mv_C^2 + 2mgr$

$$v_C = \sqrt{2g(y_A - 2r)} = \sqrt{2(9.8)(40 - 2(10))} = \boxed{19.8\,m/s}$$



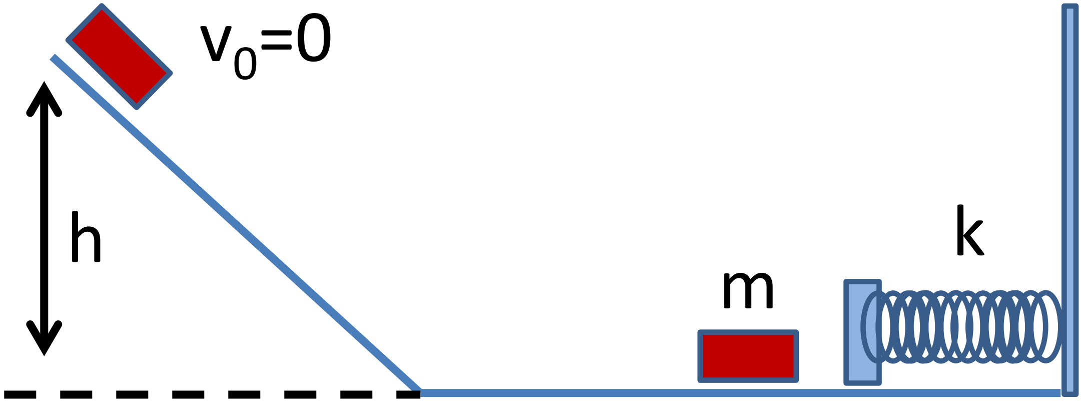
- **Example**

A block of mass m=.5 kg starts from rest at height h=2 m. The surface is frictionless and the spring constant is k=5 N/m. How far does spring compress?

$$E_0 = E_{final} \rightarrow K_0 + U_0 = K + U$$

$$0 + mgh = 0 + \frac{1}{2}kx^2 \rightarrow x = -\sqrt{\frac{2mgh}{k}} = -\sqrt{\frac{2(.5)(9.8)(2)}{5}}$$

$$= \boxed{-1.98\ m}$$



- Example

A 1.5 kg block is pressed against a spring of force constant k=2 N/m.

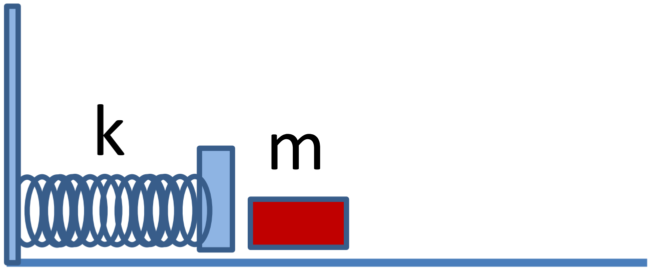
a) The spring is **compressed** by .2 m (so x= - .2 m). What is the force the spring exerts on the block?

To right

$$F = -kx = 9\left(2\frac{N}{m}\right)(-.2\, m) = \boxed{+.4\, N}$$

b) What is the change in potential energy during the compression? (Assume the block is held briefly at rest after compression.)

$$\Delta U = \frac{1}{2}\, k\, (x^2 - x_0^2) = \frac{1}{2}\left(2\frac{N}{m}\right)\left((-.2\, m)^2 - (0)^2\right) = \boxed{.04\, J}$$



Continued →

- **Example (continued)**

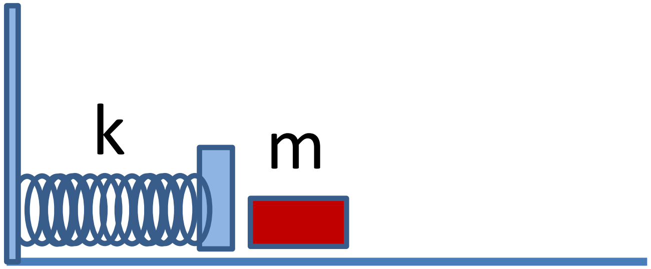
A 1.5 kg block is pressed against a spring of force constant k=2 N/m.

c) The block is now released from rest. What is its velocity just after it leaves the spring? (Ignore friction)

From part b

$$0 = \Delta K + \Delta U = (K - K_0) + (U - U_0) = \left(\frac{1}{2}\text{m}v^2 - 0\right) + (0 - .04J)$$

$$\frac{1}{2}\text{m}v^2 = .04J \quad \rightarrow \quad v = \sqrt{\frac{2(.04J)}{m}} = \sqrt{\frac{2(.04J)}{m}} = \boxed{.231\,\frac{m}{s}}$$



Continued →

• Example (continued)

A 1.5 kg block is pressed against a spring of force constant k=2 N/m.

d) Suppose the block now slides 2 m after its release before coming to rest. How much work was done by friction?

Now: v=0, v_0=.231 m/s, ΔU=0, m=1.5 kg

$$\mathrm{W} = \Delta K = (K - K_0) = \left(\frac{1}{2}\mathrm{m}v^2 - \frac{1}{2}\mathrm{m}v_0^2\right) = \frac{1}{2}(1.5)(0^2 - (.231)^2)$$

$$W = \boxed{0.040\ J}$$

e) Find the friction force and the coefficient of friction.

$$\mathrm{W} = \mathrm{Ffr\,d}\cos\theta \quad \rightarrow \quad \mathrm{F_{fr}} = \frac{W}{d\cos\theta} = \frac{-.040\ J}{(2\ m)\cos 180^\circ} = \boxed{.02\ N}$$

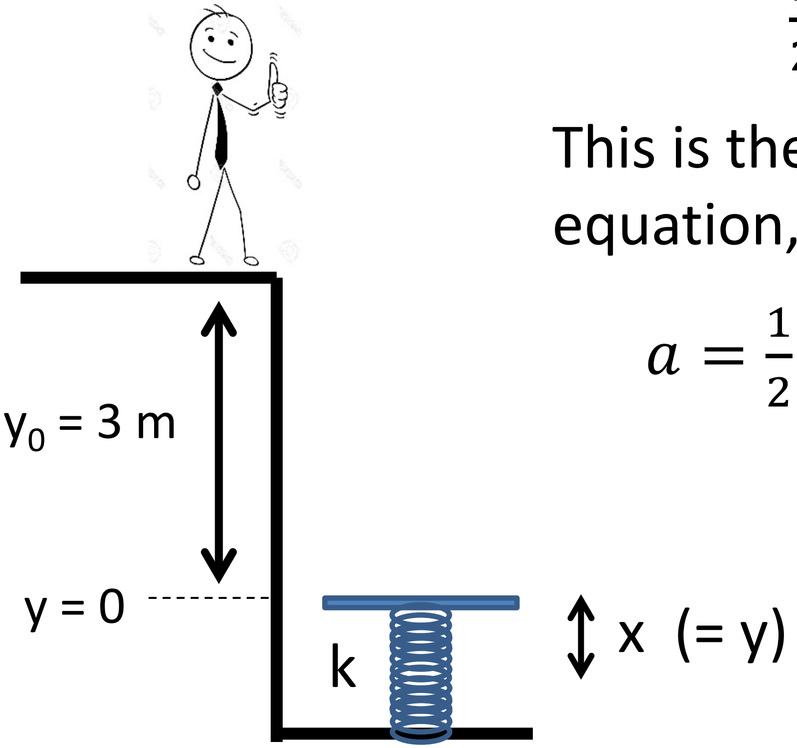
$$\mu_K = \frac{F_{fr}}{N} = \frac{F_{fr}}{mg} = \frac{0.02\ N}{(1.5\ kg)(9.8\ \frac{m}{s^2})} = \boxed{.0014}$$

F and d are in opposite directions

- **Example**
- When 50 kg man jumps onto trampoline, how much does spring compress? (k = 2000 N/m)

$$K_0 + U_0 = K + U \quad \rightarrow \quad 0 + mgy_0 = 0 + (mgx + \frac{1}{2}kx^2)$$

$$\frac{1}{2}kx^2 + mgx - mgy_0 = 0$$



This is the standard form for a quadratic equation, with:

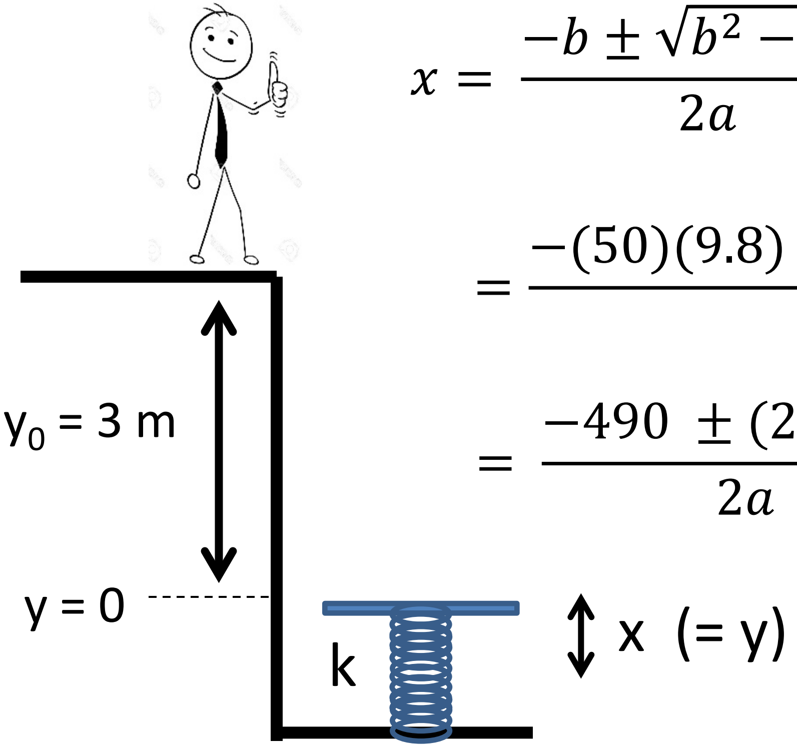
$$a = \frac{1}{2}k \qquad \mathrm{b} = +\mathrm{mg} \qquad \mathrm{c} = -\mathrm{mg}y_0$$

Solve the quadratic equation →

- **Example (continued)**
- When 50 kg man jumps onto trampoline, how much does spring compress? (k = 2000 N/m)

$a = \frac{1}{2}k \qquad \mathrm{b} = +\mathrm{mg} \qquad \mathrm{c} = -\,\mathrm{mg}y_0$

Apply the quadratic formula:



$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-mg \pm \sqrt{(mg)^2 - 4(\frac{1}{2}k)(-\,\mathrm{mg}y_0)}}{2(\frac{1}{2}k)}$$

$$= \frac{-(50)(9.8) \pm \sqrt{((50)(9.8))^2 + 2(2000)(50)(9.8)(3)}}{(2000)}$$

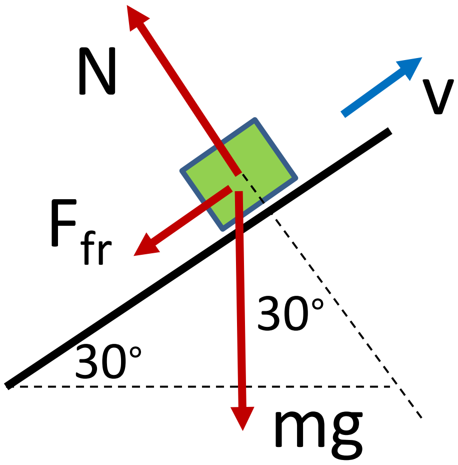
$$= \frac{-490 \ \pm (2473.9)}{2a} = \boxed{-1.48\,m}$$

Use minus sign
(must have ***compression***)

- **Example**

A 12 kg crate sits on the floor. A worker wants to slide it up a ramp (at 30° from the horizontal), into a truck. The crate is given an initial speed of 2 m/s. It slides distance d=1.6 m upward before sliding back down.

a) What is the coefficient of friction?



On the way up, forces are as shown at left.

On the way down, directions of **v** and $\mathbf{F_{fr}}$ are flipped.

From components perpendicular to ramp:

$$N = mg\cos\theta$$

So: $F_{fr} = \mu_K N = \mu_K mg\cos\theta$

Continued →

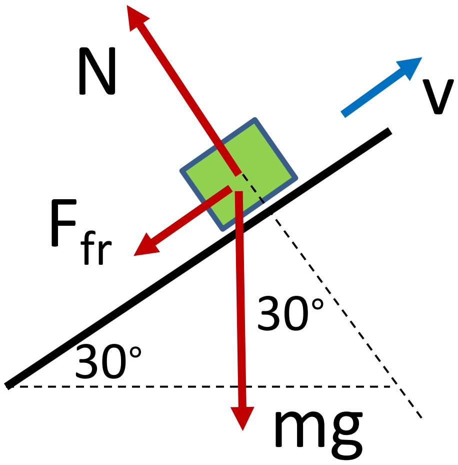
- **Example (continued)**

a) What is the coefficient of friction?

On the way up: $W = -F_{fr}\, d = -\mathrm{d}\, \mu_K \mathrm{mg} \cos\theta$

But also: $W = K - K_0 = 0 - \frac{1}{2} m v_0^2$

So: $\frac{1}{2}\cancel{m}v_0^2 = \mathrm{d}\, \mu_K \cancel{\mathrm{m}}\mathrm{g} \cos\theta$



$$\mu_K = \frac{v_0^2}{2dg\cos\theta} = \frac{(2\frac{m}{s})^2}{2(1.6)(9.8)\cos 30°}$$

$$\boxed{\mu_K = .147}$$

Continued →

- **Example (continued)**

b) How fast is the crate moving when it returns to its starting point?

The total work is twice what it was going up:

$$W = -F_{fr}\,(2d) = -2\mathrm{d}\;\mu_K \mathrm{mg}\cos\theta$$

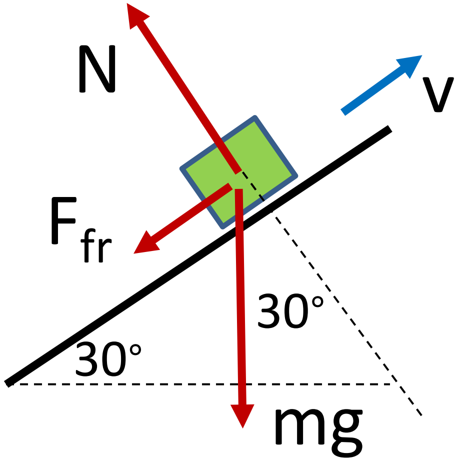
So:

$$W = K - K_0 = 0 - \frac{1}{2}mv_0^2$$

$$\frac{1}{2}\cancel{m}(v_0^2 - v^2) = 2\mathrm{d}\;\mu_K \cancel{\mathrm{m}}\mathrm{g}\cos\theta$$

$$v = \sqrt{v_0^2 - 2gd\mu_K \cos\theta}$$

$$= \sqrt{(2)^2 - 2(9.8)(1.6)(.147)\cos 30°}$$



$$\boxed{\mathrm{v} = .088\ \mathrm{m/s}}$$

Gravitational Potential Energy (general case)

- General form of gravity: $F = \dfrac{G\, m_1 m_2}{r^2}$
- Potential energy:

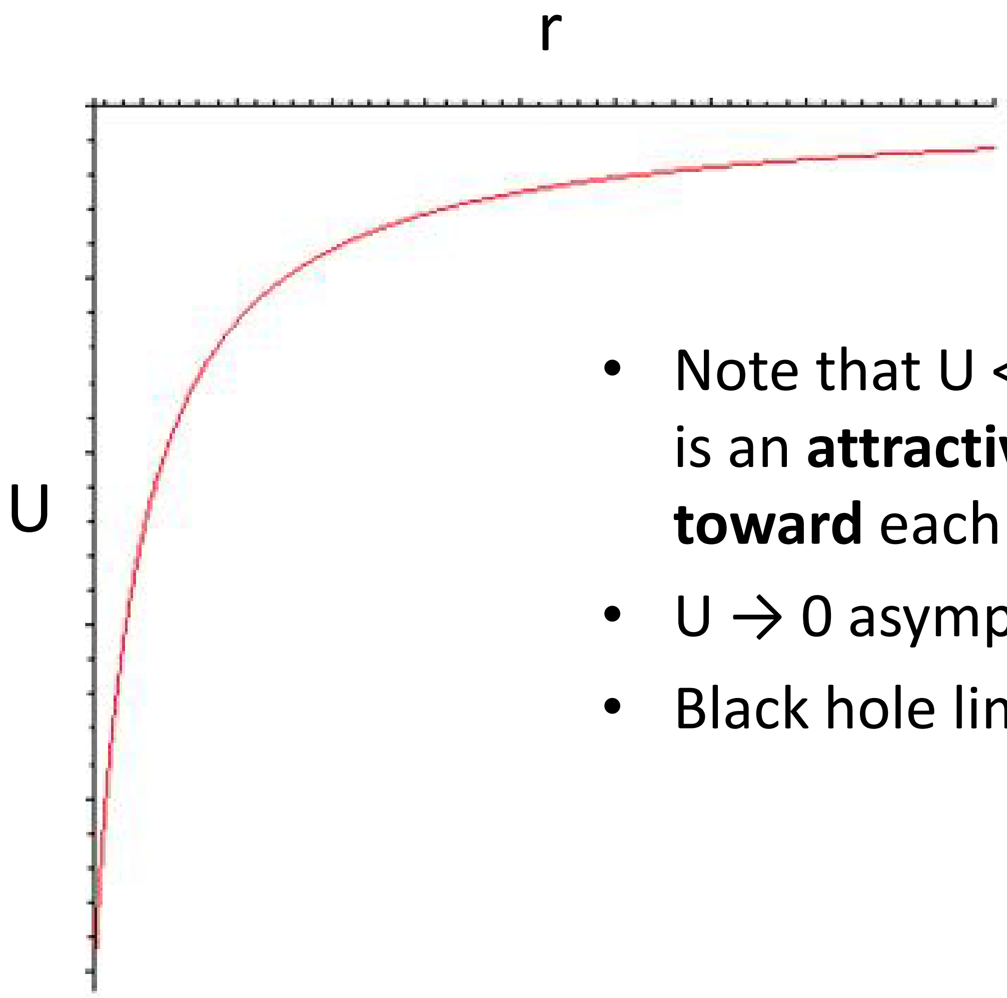
$$\Delta U = -\int_r^\infty \boldsymbol{F} \cdot \boldsymbol{dr} = -\int_r^\infty \frac{G\, m_1 m_2}{r^2}\, dr$$

$$U(r) - \cancel{U(\infty)} = +\frac{G\, m_1 m_2}{\infty} - \cancel{\frac{G\, m_1 m_2}{r}}$$

Choose reference point so that U(∞) = 0

$$\boxed{U(r) = -\frac{G\, m_1 m_2}{r}}$$

Gravitational Potential Energy



- Note that $U < 0$, implying that gravity is an **attractive** force. It pulls masses **toward** each other.
- $U \rightarrow 0$ asymptotically as $r \rightarrow \infty$
- Black hole limit: $U \rightarrow -\infty$ as $r \rightarrow 0$

- **Example**

What is the minimum speed a projectile must have to leave the Earth’s gravitational pull?

U=0 at large distance

K=0 is minimum K left over at large distance

$$U_0 + K_0 = U + K \quad \rightarrow \quad -\frac{GM_E m}{R_E} + \frac{1}{2}mv^2 \geq 0 + 0$$

$$v = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{\frac{2(6.67 \times 10^{-11})(5.98 \times 10^{24})}{6.37 \times 10^{6}}}$$

$$= 1.12 \times \frac{10^4 m}{s} = \boxed{11.2 \ \frac{km}{s}}$$

- The speed just found is called Earth’s **escape velocity**:

$$\boxed{v_{esc} = \sqrt{\frac{2GM_E}{R_E}}}$$

- For comparison:
  - Sun: $v_{esc} = 618\ \frac{km}{s}$
  - Earth: $v_{esc} = 11.2\ \frac{km}{s}$
  - Moon: $v_{esc} = 2.38\ \frac{km}{s}$

- **Example**

To what radius must the mass of the Earth be compressed in order to create a black hole?

A black hole occurs when $v_{esc} \geq c$, where $c = 3 \times 10^8 \frac{m}{s}$ is the speed of light.

$$v_{esc} = \sqrt{\frac{2GM_E}{R}} = c$$

Solve for R:

$$R = \frac{2GM_E}{c^2} = \frac{2(6.67\times10^{-11})(5.98\times10^{24})}{(3\times10^8)^2} = .0089\text{ m}$$

$$\boxed{\mathrm{R} = 8.9\text{ mm}}$$

- **NOTE:**
- We now have two formulas for the gravitational potential energy from the Earth on another mass:

$$U_G = -G\,\frac{M_E\,m}{r} \qquad \text{and} \qquad U_G = mgy$$

- These two formulas use different reference points to define U=0, and so **they are inconsistent** with each other.
- As a result, if you are comparing finding the potential energy at two different points, consistency requires that you use the more general formula at **both** points, even if one point is at the Earth's surface.
- You **cannot** mix and match the two formulas. See the next example.

- **Example**

A rocket is launched at a speed of $v_0 = 8\ \frac{km}{s}$.

How high an altitude will it reach?

$$U_0 + K_0 = U + K \quad \rightarrow \quad -\frac{GM_E m}{R_E} + \frac{1}{2} m v_0^2 = -\frac{GM_E m}{r} + 0$$

$$r = \frac{GM_E}{\frac{GM_E}{R_E} - \frac{1}{2} v_0^2} = \frac{(6.67 \times 10^{-11})(5.98 \times 10^{24})}{\frac{(6.67 \times 10^{-11})(5.98 \times 10^{24})}{6.37 \times 10^6} - \frac{1}{2}(8 \times 10^3\ \frac{m}{s})^2}$$

$$= 1.3 \times 10^7\ m = 1300\ km$$

This is distance from Earth's center. **Altitude** is:

$$h = r - R_E = 13000\ km\ - 6370\ km = \boxed{6730\ km}$$

CH. 9 – LINEAR MOMENTUM

Linear momentum

- Recall: when we went from working with acceleration and force (vectors) to using energy (a scalar), we lost information about directions.
- The goal is to restore directional.
- We want to introduce a new **vector** quantity that is convenient to use in conjunction with energy.
- Our choice will be **momentum:** combined use of energy and momentum conservation laws will make it possible to easily solve many problems that would otherwise be very difficult.

Linear momentum

- Momentum:

$$\boxed{\boldsymbol{p} = m\boldsymbol{v}}$$

- Notice that $\boldsymbol{p}$ is a **vector**. It has components and a direction.
- Units: $\frac{kg \cdot m}{s}$

Newton’s second law

- One reason why momentum is useful is that it is closely related to force.
- We can generalize Newton’s second law to:

$$\boxed{\Sigma \boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}}$$

- This reduces back to our old version when the mass of the object is constant (next slide)

Linear momentum

- Using the product rule of calculus:

$$\Sigma \boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = \frac{d(m\boldsymbol{v})}{dt} = m\frac{d\boldsymbol{v}}{dt} + \frac{dm}{dt}\boldsymbol{v}$$

$$\Sigma \boldsymbol{F} = m\boldsymbol{a} + \frac{dm}{dt}\boldsymbol{v}$$

- So when the mass is constant the second term vanishes and we just get back $\Sigma \boldsymbol{F} = m\boldsymbol{a}$.
- But now we see that if the mass is changing, we can have a net force even without acceleration.

Impulse

- The **impulse** is the momentum change $\Delta \boldsymbol{p}$ that a force imparts to an object.
- If the force acts for some time period Δt, then the impulse is:

$$\Delta \boldsymbol{p} = \int_0^{\Delta t} \boldsymbol{F}(t)dt$$

- This is obtained simply by integrating both sides of Newton's 2nd law.
- If the force is constant (not a function of time), the impulse simplifies to: $\Delta \boldsymbol{p} = \boldsymbol{F}\Delta t$

Isolated Systems

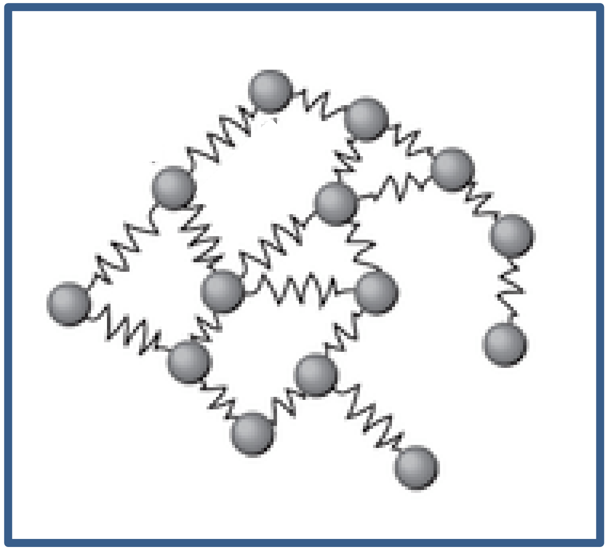
- An isolated system is a collection of objects with $\Sigma \boldsymbol{F}$=0.
- This occurs if either there are no external forces, or the external forces cancel.
- As a result the total momentum of the system is constant, by Newton's 2nd law.
- In other words, the total momentum of the system is conserved:

$$\boldsymbol{p}_{total} = constant$$

Isolated Systems

- Notice that although the total momentum of the system is constant, individual constituents within the system can exchange momentum with each other.
- So the momenta of these subsystems are not conserved, but the **total** (the sum over all of the subsystems) is.

Box full of masses connected by springs: Each mass can exchange momentum with others, but the **total** momentum of the entire system is constant



Conservation of Linear Momentum

- We will write the conservation of momentum in this form:

$$\boxed{\sum_{initial} \boldsymbol{p} = \sum_{final} \boldsymbol{p}'}$$

- The sum is over all objects in the system.
- The left side involves the **initial** momenta (before a collision, for example), the right side involves the final momenta.
- The apostrophe is called a ***prime***.
- Henceforth, primed variables will be final values, and unprimed will be initial.

Conservation of Linear Momentum

- Recall that momentum is a vector. So the vector equation

$$\sum_{initial} \boldsymbol{p} = \sum_{final} \boldsymbol{p}'$$

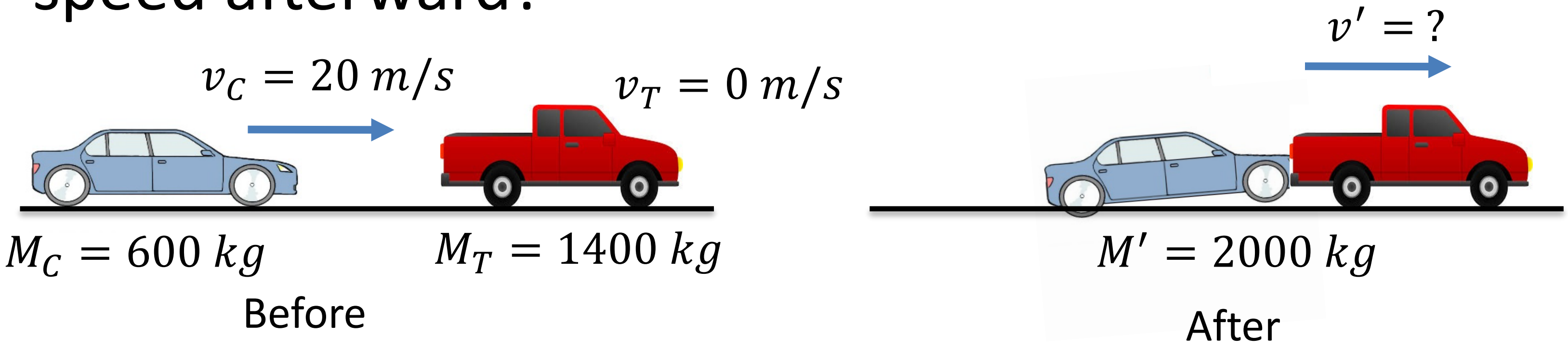
is really shorthand for several equations, one for each component:

$$\sum_{initial} \boldsymbol{p}_x = \sum_{final} \boldsymbol{p}'_x \qquad \sum_{initial} \boldsymbol{p}_y = \sum_{final} \boldsymbol{p}'_y \qquad \sum_{initial} \boldsymbol{p}_z = \sum_{final} \boldsymbol{p}'_z$$

Momentum conservation **must** be applied to each component **separately**.

Example 1

- A car of mass 600 kg traveling at 20 m/s collides with a stationary truck of mass 1400 kg. The two are locked together after the collision. What is their combined speed afterward?


$v_C = 20\ m/s$
$v_T = 0\ m/s$
$v' = ?$
$M_C = 600\ kg$
$M_T = 1400\ kg$
$M' = 2000\ kg$
Before
After

- If we ignore friction, then the sum of **external** forces is zero. (Gravity and normal force cancel).
- Only **internal** forces **between** the cars act. So total momentum of the 2-car system is conserved.

Example 1 (continued)

- The problem is one-dimensional (in the x-direction), so only x components are needed.

$$\sum_{\boldsymbol{initial}} \boldsymbol{p_x} = \sum_{\boldsymbol{final}} \boldsymbol{p'_x} \qquad \rightarrow \qquad M_C v_C + M_T v_T = M' v'$$

$$v' = \frac{M_C v_C + M_T v_T}{M'} = \frac{(600\,kg)\left(20\,\frac{m}{s}\right) + 0}{2000\,kg}$$

$$= \boxed{+6\,m/s}$$

To right

Example 2

- A toy dart gun has a mass of .3 kg. It fires a projectile of mass 60 g (.06 kg) at a speed of 50 m/s.
- What is the recoil velocity of the toy gun?
- Given: $m_G = .3\,kg\,, \quad m_d = .06\,kg, \quad v_G = v_d = 0$

$$v'_d = +50\,\frac{m}{s}, \quad v'_G = ?$$

$$\sum_{initial} \boldsymbol{p}_x = \sum_{final} \boldsymbol{p}'_x \quad \rightarrow \quad 0 = m'_G v'_G + m'_d v'_d$$

- So: $v'_G = -\frac{m'_d v'_d}{m'_G} = -\frac{(.06)(50)}{.3} = \boxed{-10\,m/s}$

Gun recoils backward (to left)

Example 3

- A rocket is propelled by a stream of exhaust expelled at a rate of 100 kg/s. It's velocity is v=50m/s (assumed constant).

(a) What is the size of the force propelling the rocket forward?

a=0, but force is nonzero because of changing mass

$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta m}{\Delta t} v = \left(100 \frac{kg}{s}\right)\left(50 \ \frac{m}{s}\right) = \quad 5000 \ N$$

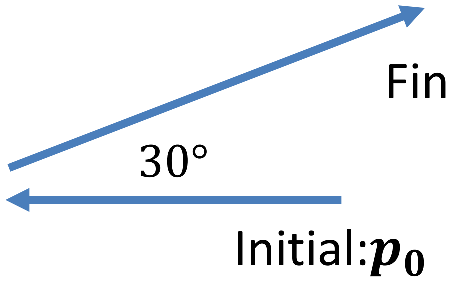
(b) By how much does the rocket's momentum change during a 2 second interval.

$$\Delta p = -\left(\frac{\Delta p}{\Delta t}\right) \Delta t = -F\Delta t = -(5000 \ N)(2 \ s) = \boxed{-10{,}000 \frac{kg \ m}{s}}$$

Rocket loses momentum (exhaust gains it)

Example 4

- A .25 kg baseball is moving horizontally at 40 m/s. After being struck by a bat, its new speed is 60 m/s, at 30° above the horizontal. The bat and ball were in contact for .005 s. Find the average force exerted by the bat.


Final: $\boldsymbol{p}$
30°
Initial: $\boldsymbol{p_0}$

Given: $m_{ball} = .25kg,\ v = 40\frac{m}{s}\ (to\ left),\ \Delta t = .005s$

$$v' = 60\frac{m}{s}\ (at\ 30°)$$

Example 4 (continued)

- Components of initial and final momenta:

$$p_x = mv_x = (.25\,kg)\left(-40\,\frac{m}{s}\right) = -10\,kg\,m/s$$

$$p_y = mv_y = 0$$

$$p_x' = mv' \cos\theta = (.25)(60)\cos 30° = 12.99\,\frac{kg \cdot m}{s}$$

$$p_y' = mv' \sin\theta = (.25)(60)\sin 30° = 7.5\,\frac{kg \cdot m}{s}$$

- Now find force ➡

Example 4 (continued)

- Components of force:

$$F_x = \frac{\Delta p_x}{\Delta t} = \frac{p'_x - p_x}{\Delta t} = \frac{12.99 - (-10)}{.005} = \boxed{4598\ N}$$

$$F_y = \frac{\Delta p_y}{\Delta t} = \frac{p'_y - p_y}{\Delta t} = \frac{7.5 - (0)}{.005} = \boxed{1500\ N}$$

- Magnitude and direction:

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(4598)^2 + (1500)^2} = \boxed{4836.5\ N}$$

$$\theta = \tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\frac{1500}{4598} = \boxed{18.06\,°}$$

Collisions

- **Collision:** Two or more objects move toward each other, collide, and move away.
- The objects exchange energy and momentum, so their velocities change.
- Goal is to find final velocities.
- We will only consider collisions of two objects.
- We will also assume that the collision occurs rapidly enough the impulse from any external force is negligible.
- The last assumption implies that **momentum is conserved** in all the collisions we will consider.

Types of collisions

- Although momentum is always conserved in collisions, energy may not be.
- **Elastic collision** - Energy is conserved, so we have two conservation laws:

$$\sum_{initial} \boldsymbol{p} = \sum_{final} \boldsymbol{p}' \quad \boldsymbol{and} \quad \sum_{initial} \boldsymbol{K} = \sum_{final} \boldsymbol{K}'$$

Potential energy usually does't change during a rapid collision, so only kinetic energy needs to be included.

- **Inelastic collision** – Energy is lost, so there is only one conservation law available: $$\sum_{initial} \boldsymbol{p} = \sum_{final} \boldsymbol{p}'$$
- In an inelastic collision, the energy may be carried away by heat, sound waves, radiation, etc.

Types of Collisions

- Recall that the kinetic energy is quadratic in the speeds, so in the elastic case you will get two answers for the final velocity. One will be unphysical and must be dropped.
- Special case of inelastic collisions: a **totally inelastic** collision is one where the objects stick together. So the two final velocities have to be equal: $v_1' = v_2'$ (so just call them v')
- **Be careful**: all **totally inelastic** collisions are **inelastic**. But the reverse is **not** true - not all inelastic collisions are totally inelastic.

Shortcut method

- There is a special case for elastic collisions as well.
- For elastic collisions, p and E are conserved:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

Momentum conservation

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

Energy conservation

- If a collision is both (i) **one-dimensional** and (ii) **elastic**, then these can be replaced by:

$$\boxed{\begin{gathered} m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \\ v_1 + v_1' = v_2 + v_2' \end{gathered}}$$

- The **quadratic** energy conservation equation is replaced by the **linear** “shortcut equation”.

Proof of Shortcut Method

- Start from the energy and momentum conservation equations in one dimension.

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \qquad \textbf{A}$$

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \qquad \textbf{B}$$

- Cancel the $\frac{1}{2}$'s and rearrange these as follows:

$$m_1 (v_1 - v_1') = m_2 (v_2' - v_2) \qquad \textbf{C}$$

$$m_1 (v_1^2 - v_1'^2) = m_2 (v_2'^2 - v_2^2) \qquad \textbf{D}$$

- Equation **D** factors:

$$m_1 (v_1 - v_1')(v_1 + v_1') = m_2 (v_2' - v_2)(v_2' + v_2) \qquad \textbf{E}$$

Proof of Shortcut Method

- But Eq. **E** can be simplified using **D**:

$$\underbrace{m_1(v_1-v_1')}_{\text{Left side of }\mathbf{C}}(v_1+v_1') = \underbrace{m_2\,(v_2' - v_2)}_{\text{Right side of }\mathbf{C}}(v_2' + v_2)$$

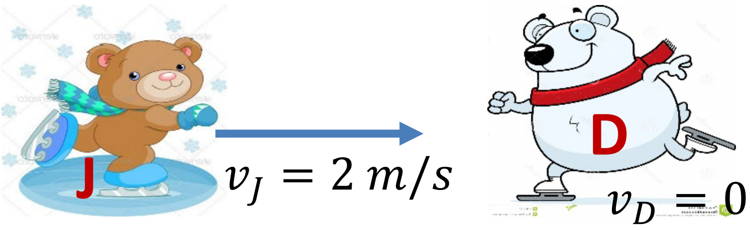
- So the terms in red cancel, leaving:

$$v_1 + v_1' = v_2' + v_2$$

- But this is just the shortcut formula. So the result is proven.
- **IMPORTANT:** The shortcut method can **only** be used if **both** of the following are true: (i) the system is one-dimensional and (ii) the collision is elastic.

Example 5

- Two bears (Joe and Donald) are ice-skating when they collide. They undergo an **elastic** collision as their bellies bounce off each other.
- Suppose Donald is 20% heavier than Joe: $m_D = 1.2m_J$
- Initial speeds: $v_J = 2\ m/s$ and $v_D = 0$.
- If Joe bounces back along his initial line of motion (collision is one dimension), then find the final speeds, v_J' and v_D'.
- The conditions are satisfied that allow us to use the shortcut method.



Example 5 (continued)

- Using shortcut method:

$$m_J v_J + m_D \cancel{v_D} = m_J v_J' + m_D v_D' \quad \rightarrow \quad m_J v_J = m_J v_J' + m_D v_D' \quad \textbf{A}$$

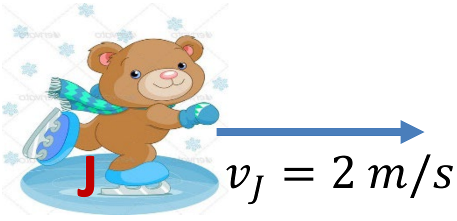
$$v_J + v_J' = \cancel{v_D} + v_D' \quad \rightarrow \quad v_J + v_J' = v_D' \quad \textbf{B}$$

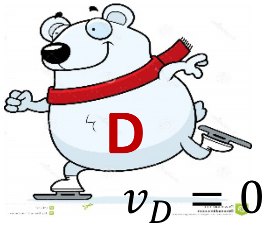
- But using $m_D = 1.2 m_J$, **A** becomes: $v_J = v_J' + 1.2 v_D'$ **C**

- Plug **B** into **C** to eliminate v_D':

$$v_J = v_J' + 1.2\left(v_J + v_J'\right) \quad \rightarrow \quad -0.2 v_J = 2.2 v_J'$$

Solve for v_J'





Example 5 (continued)

- Solve for v_J':

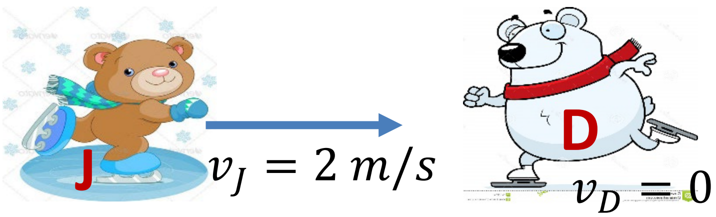
$$v_J' = \frac{-0.2v_J}{2.2} = \frac{-0.2(2)}{2.2} = \boxed{-0.182\,m/s}$$

(to left)

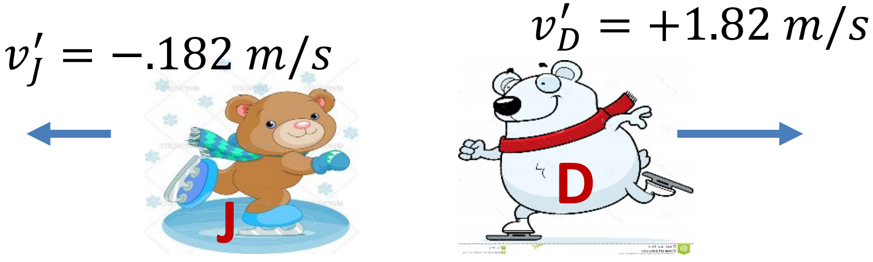
- Substitute back into **B**:

(to right)

$$v_D' = v_J + v_J' = 2 + (-0.182) = \boxed{+1.82\,m/s}$$


J
$v_J = 2\,m/s$
D
$v_D = 0$

Before


$v_J' = -.182\,m/s$
$v_D' = +1.82\,m/s$
J
D

After

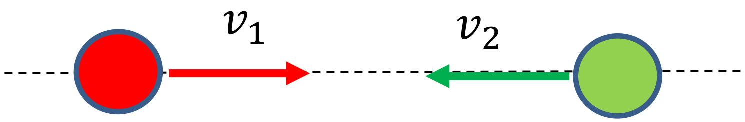
Disclaimers: No bears were harmed in the construction of this example.

Example 6

- Two billiard balls of equal mass undergo an elastic collision.
- Initial velocities: $v_1 = +.3\, m/s, \quad v_2 = -.2\, m/s$
- They bounce back along the same line.
- What are the final velocities?
- Again use the shortcut:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \xrightarrow{m_1 = m_2} v_1 + v_2 = v_1' + v_2' \quad \textbf{A}$$

$$v_1 + v_1' = v_2 + v_2' \quad \textbf{B}$$

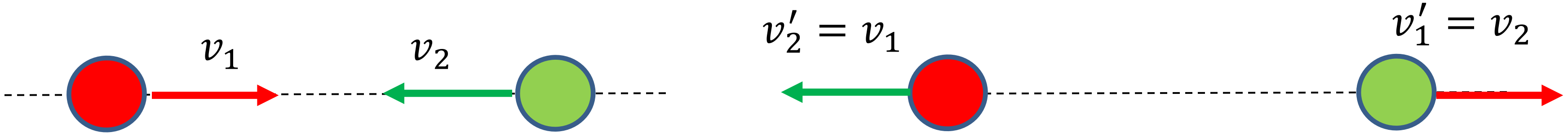


Example 6 (Continued)

- From **A**: $v_1' = v_1 + v_2 - v_2'$ **C**
- Plug this into **B**: $v_1 + (v_1 + \cancel{v_2} - v_2') = \cancel{v_2} + v_2'$
- So: $2v_1 = 2v_2'$ → $v_2' = v_1 = \boxed{.3\ m/s}$
- Plug this back into **C** :

$$v_1' = v_1 + v_2 - v_2' = .3 - .2 - .3 = \boxed{-.2\ m/s}$$

- Notice that the two objects have simply exchanged velocities. (This only happens when the masses are equal and the collision is elastic.)

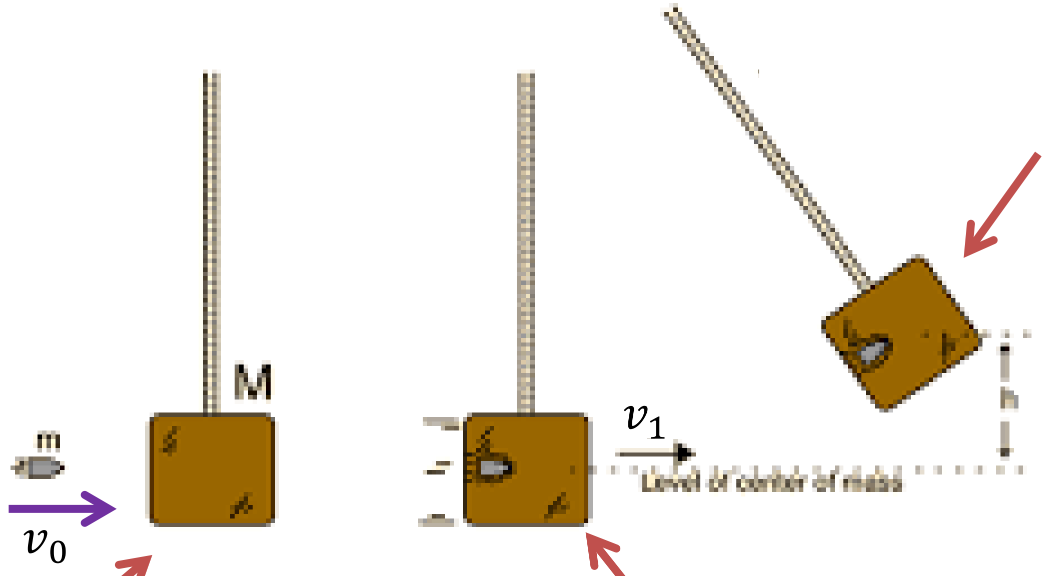


Before

After

Example 7

- A bullet of mass $m = 6\ g = .006\ kg$ is fired into a wooden block of mass $M = .4\ kg$ hanging from a string. The block rises a distance of $h = .14\ m$.



Time $\boldsymbol{t_2}$: At maximum height

Time $\boldsymbol{t_0}$: just before collision

Time $\boldsymbol{t_1}$: just after collision. Block now moving, but height has not had time to change yet

Example 7 (continued)

- Goal is to find initial speed of bullet from measured height h.
- Will work backward in two steps:
  - Start at t_2, where we essentially now everything, and work back to t_1 to find speed just after collision.
  - Work backward from t_1 to t_0 to find initial speed.
- The key is to determine what is conserved at each step.

Example 7 (continued)

- Step 1: Find speed v_1 just after collision.
- During the upward swing (between t_1 and t_2), the only forces acting are tension and gravity.
- These are conservative forces.
- So energy is conserved. Apply this fact to find v_1.

Notice that momentum is **not** conserved. Momentum is nonzero at t_1 but will be zero at t_2.

Example 7 (continued)

- Energy conservation between times t_1 and t_2:

$$E_1 = E_2 \quad \rightarrow \quad \frac{1}{2}(m+M)v_1^2 = (m+M)gh$$

Both masses move together.
At t_1, energy is all kinetic.

At t_2, energy is
all potential.

$$v_1 = \sqrt{2gh} = \sqrt{2(9.8)(.14)} = \boxed{1.66\, m/s}$$

Example 7 (continued)

- Step 2: Find speed v_0 just before collision.
- Bullet embeds in wood, so collision is totally inelastic → Energy **is not** conserved.
- But initially the tension and gravity cancel. Sum of external forces is zero → Momentum **is** conserved.
- So apply momentum conservation in the x direction

Example 7 (continued)

- $\Sigma\, p = \Sigma\, p' \quad \rightarrow \quad mv_0 + 0 = (m + M)v_1$
- Solve for v_1:

$$v_0 = \left(\frac{m + M}{m}\right) v_1 = \left(\frac{.006 + .4}{.006}\right)(1.66)$$

$$= \boxed{112.3\ m/s}$$

(Initial bullet speed)

Example 8

- At an intersection, a car of mass $m_c = 1500\ kg$ (travelling east at $v_c = 25\ m/s$) and a van of mass $m_v = 2500\ kg$ (travelling north at $v_v = 20\ m/s$) collide.
- They vehicles stick together after the collision – the collision is **totally inelastic**.
- Since they stick together, the car and the van have the **same** final velocity.
- Find the magnitude and direction of the final velocity.

Example 8 (continued)

- The collision is inelastic, so energy is lost.
- But momentum is conserved:

$$\Sigma\, p_x = \Sigma\, p'_x \quad \rightarrow \quad m_C v_C + 0 = (m_C + m_V)v' \cos\theta$$ **A**

$$\Sigma\, p_y = \Sigma\, p'_y \quad \rightarrow \quad 0 + m_V v_V = (m_C + m_V)v' \, sin\, \theta$$ **B**

- Divide Eq. **B** by **A**: $\dfrac{m_V v_V}{m_C v_C} = \tan\theta$

$$\theta = \tan^{-1}\frac{m_V v_V}{m_C v_C} = \tan^{-1}\frac{(2500)(20)}{(1500)(25)} = \boxed{53.1°}$$

- Solve **A** for v':

$$v' = \frac{m_C v_C}{(m_C + m_V)\ \cos\theta} = \frac{(1500)(25)}{1500 + 2500)\ \cos 52.1°} = \boxed{15.6\, m/s}$$

CH. 21- ELECTRIC CHARGE & FIELD

BASIC FORCES:
STRONG NUCLEAR
ELECTROMAGNETIC
WEAK NUCLEAR
GRAVITY

↑ INCREASING STRENGTH

ELECTROMAGNETISM IS RESPONSIBLE FOR:

- NORMAL FORCES
- THE FORCES THAT HOLD ATOMS TOGETHER IN MOLECULES & THAT HOLD MOLECULES TOGETHER IN SOLIDS & LIQUIDS
- CHEMICAL REACTIONS
- PRODUCTION OF LIGHT, RADIO WAVES, MICROWAVES

THE SUBJECT DIVIDES INTO 3 PARTS:

1. ELECTRICITY (DUE TO STATIC CHARGES)
2. MAGNETISM (DUE TO MOVING CHARGES)
3. ELECTROMAGNETIC RADIATION (DUE TO ACCELERATING CHARGES)

CHARGE: JUST AS GRAVITY IS CREATED BY MASS, ELECTROMAGNETISM IS CREATED BY **CHARGE**.

CHARGES WILL BE DENOTED BY q OR Q.

UNITS OF CHARGE: **COULOMBS** (C)

UNLIKE MASS, CHARGE CAN BE **POSITIVE** OR **NEGATIVE**:

$$\left.\begin{array}{l}\text{ELECTRON: } q = -e \\ \text{PROTON: } q = +e\end{array}\right\} \boxed{e = 1.6 \times 10^{-19}\,\text{C}}$$

PROTONS & NEUTRONS ARE BUILT FROM **QUARKS**, WITH CHARGES OF $\pm\frac{1}{3}$, $\pm\frac{2}{3}$.

$$\boxed{\text{TOTAL CHARGE IS ALWAYS CONSERVED}}$$

CURRENT = RATE OF FLOW OF CHARGE: $\boxed{I = \dfrac{dq}{dt}}$

NORMALLY, CARRIERS OF CURRENT ARE ELECTRONS.

CAN CLASSIFY MATERIALS BY HOW WELL THEY CARRY CURRENTS:

1. **CONDUCTORS**: ELECTRONS FLOW EASILY WITH LITTLE RESISTANCE (METALS, FOR EXAMPLE)
2. **INSULATOR** = **DIELECTRIC**: THE ELECTRONS ARE LARGELY CONFINED ATOMS & HAVE LITTLE MOBILITY (GLASS, PLASTIC)
3. **SUPERCONDUCTORS**: PERFECT CONDUCTOR WITH **NO** RESISTANCE AT LOW TEMPERATURES. (THESE ARE USUALLY INSULATORS AT ROOM TEMP.)

ELECTROSTATICS: CHARGES ARE AT REST (OR MOVING ONLY AT LOW SPEEDS) ⇒ CHARGES PRODUCE ELECTRIC FIELD ONLY (MAGNETIC FIELDS ARE NEGLIGIBLE)

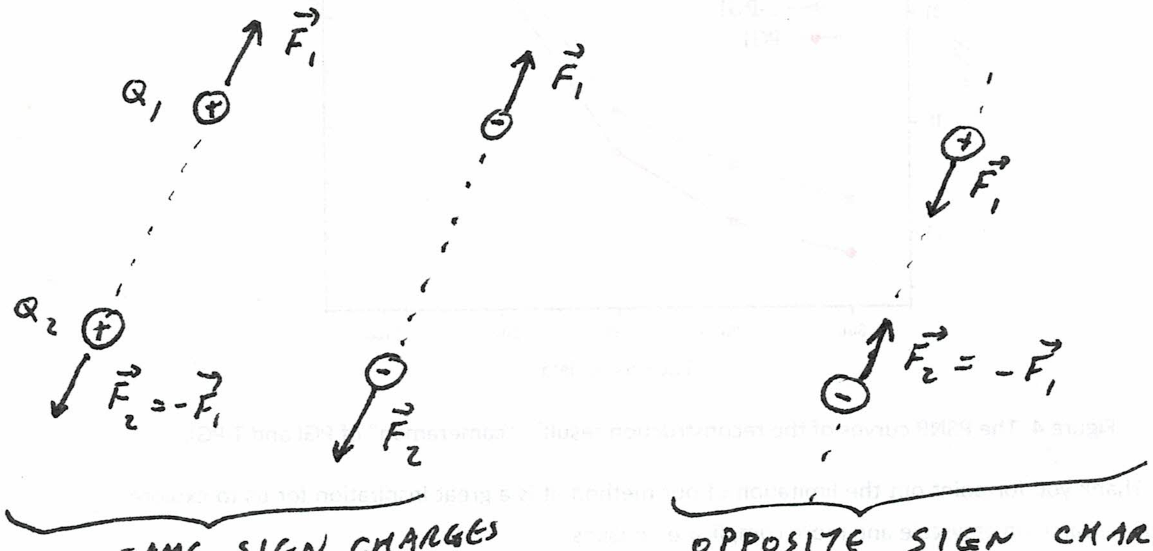
TWO CHARGES WILL EXERT ELECTROSTATIC FORCES ON EACH OTHER.

CALL THE CHARGES Q_1 & Q_2. LET r = DISTANCE BETWEEN THEM.

IF THE CHARGES ARE THE SAME SIGN (BOTH POSITIVE OR BOTH NEGATIVE), THEY REPEL.

IF THE CHARGES ARE OPPOSITE SIGN, THEY ATTRACT.

EITHER WAY, THE FORCES ARE EQUAL & OPPOSITE (NEWTON'S 3rd LAW):

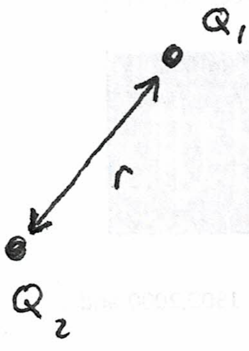


SAME SIGN CHARGES REPEL

OPPOSITE SIGN CHARGES ATTRACT

WHAT IS THE MAGNITUDE OF THIS FORCE? →

THE MAGNITUDE OF THESE ELECTROSTATIC FORCES IS GIVEN BY COULOMB'S LAW:



$$\boxed{\text{COULOMB'S LAW:} \quad F = k\frac{Q_1 Q_2}{r^2}}$$

$k = 9 \times 10^9 \frac{Nm^2}{C^2}$ = ELECTROSTATIC CONSTANT

WARNING: DON'T CONFUSE THIS CONSTANT WITH BOLTZMANN'S CONSTANT (ALSO WRITTEN AS k) ON THE FORMULA SHEET.

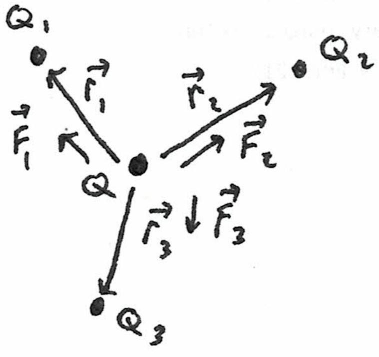
ALTERNATE FORM OF COULOMB'S LAW:

$$\boxed{F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 r^2}}$$

$\varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2}$ = PERMITTIVITY OF FREE SPACE

THE 2 CONSTANTS ARE RELATED BY: $k = \frac{1}{4\pi\varepsilon_0}$

IF THERE ARE MORE THAN 2 CHARGES, MUST FIND FORCE BETWEEN EACH PAIR & ADD THE FORCES AS VECTORS.



TOTAL FORCE ON Q:

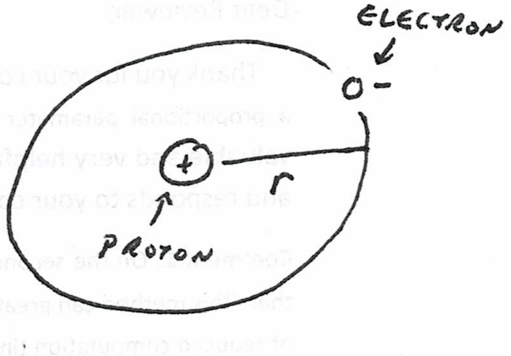
$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

EXAMPLES

① a) HYDROGEN ATOM IN ITS GROUND STATE:

LOWEST ENERGY STATE

AVERAGE RADIUS OF ELECTRON ORBIT:



$$r = 5.3\times10^{-11}\,m$$

CHARGES:

PROTON – $q_1 = 1.6\times10^{-19}\,C$

ELECTRON – $q_2 = -1.6\times10^{-19}\,C$

MAGNITUDE OF FORCE ON ELECTRON?

$$F = \left|\frac{kq_1q_2}{r^2}\right| = \frac{(9\times10^{9})(1.6\times10^{-19})^2}{(5.3\times10^{-11})^2}$$

$$= \boxed{8.2\times10^{-8}\,N}$$

ABSOLUTE VALUE BECAUSE MAGNITUDE OF FORCE IS POSITIVE

b) PRETENDING THAT THE ELECTRON ORBIT IS CIRCULAR, WHAT IS THE ELECTRON'S SPEED?

THE ELECTROSTATIC FORCE IS PROVIDING THE CENTRIPETAL FORCE:

$$F = m\frac{v^2}{r} \rightarrow v = \sqrt{\frac{Fr}{m}} = \sqrt{\frac{(8.2\times10^{-8}\,N)(5.3\times10^{-11}\,m)}{9.11\times10^{-31}\,kg}}$$

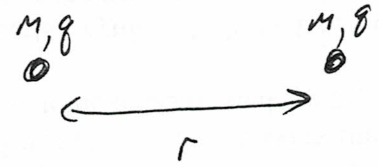
MASS OF ELECTRON

$$= \boxed{2.2\times10^{6}\,\frac{m}{s}}$$

(ABOUT 2/3 OF SPEED OF LIGHT)

② EXAMPLE: TWO IDENTICAL OBJECTS (SAME MASS m & SAME CHARGE q) ARE SEPERATED BY DISTANCE $r = 1m$.

a) SUPPOSE $m = 10^6$ kg. WHAT MUST q BE FOR ELECTRIC FORCE TO CANCEL GRAVITATIONAL FORCE?



$$F_G = F_E \rightarrow G\frac{m^2}{r^2} = k\frac{q^2}{r^2}$$

$$q = \pm\sqrt{\frac{G}{k}m^2} = \pm\sqrt{\frac{G}{k}}\cdot m = \pm\sqrt{\frac{6.67\times10^{-11}}{9\times10^{9}}}\cdot 10^6$$

$$= \boxed{8.6\times10^{-5}\ C}$$

b) SUPPOSE THIS CHARGE IS DUE TO EXCESS ELECTRONS. HOW MANY ELECTRONS NEEDED?

$$N = \left|\frac{\text{TOTAL CHARGE}}{\text{ELECTRON CHARGE}}\right| = \frac{8.6\times10^{-5}}{1.6\times10^{-19}}$$

$$= \boxed{5.4\times10^{9}\ \text{ELECTRONS}}$$

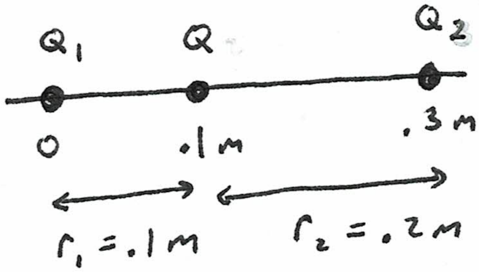
c) WHAT IS THE MASS OF THESE ELECTRONS?

$$M_{\text{ELECTRONS}} = N \times \left(\begin{matrix}\text{MASS PER}\\ \text{ELECTRON}\end{matrix}\right)$$

$$= (5.4\times10^{9})\,(9.11\times10^{-31}\ \text{kg})$$

$$= \boxed{4.9\times10^{-21}\ \text{kg}}$$

③



$Q_1 = 2\mu C$, $Q_2 = 4\mu C$, $Q = 3\mu C$
(μ = MICRO $= 10^{-6}$)

FORCE ON Q ?

FROM Q_1: $|F_1| = k\dfrac{QQ_1}{r_1^2} = 5.4\,N$ (TO RIGHT, $\theta = 0°$)

FROM Q_2: $|F_2| = k\dfrac{QQ_2}{r_2^2} = 2.7\,N$ (TO LEFT, $\theta = 180°$)

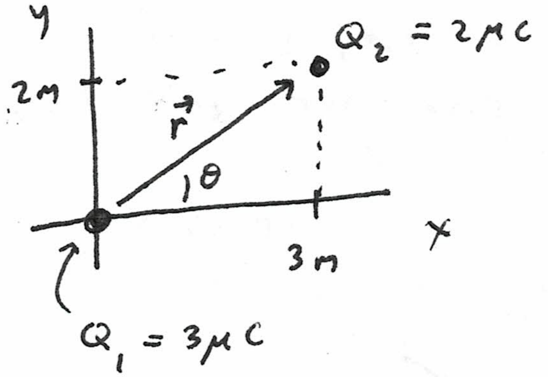
TOTAL FORCE ON Q:

$$F = |F_1| \underset{\substack{\uparrow \\ \text{TO LEFT}}}{-} |F_2| = 5.4 - 2.7 = \boxed{\underset{\substack{\uparrow \\ \text{TO RIGHT} \\ (\theta = 0°)}}{+}\ 2.7\,N}$$

HERE, THE PROBLEM IS ONE-DIMENSIONAL, SO DIRECTIONS ARE GIVEN BY ± SIGNS.

NOW TRY 2-DIMENSIONAL EXAMPLES →

④


y
2m
$Q_2 = 2\mu C$
$\vec{r}$
θ
3m
x
$Q_1 = 3\mu C$

a) FIND MAGNITUDE OF FORCE ON EACH CHARGE,

DISTANCE BETWEEN THEM:

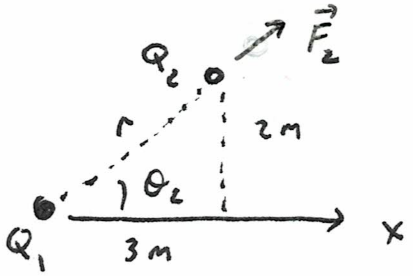
$r = \sqrt{2^2 + 3^2} = \sqrt{13}$ (IN METERS)

FORCE:

$$F_1 = F_2 = \frac{kQ_1Q_2}{r^2} = \frac{(9\times10^9)(3\times10^{-6})(2\times10^{-6})}{(\sqrt{13})^2}$$

$$= \boxed{.00415 \text{ N}}$$ (MAGNITUDE)

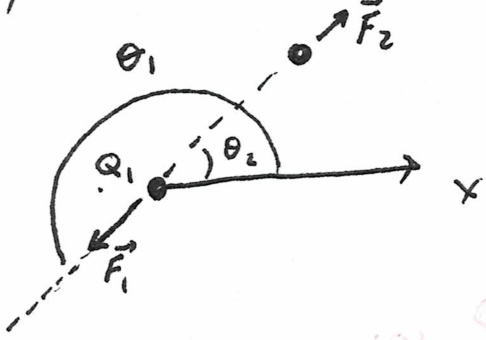
b) DIRECTION OF FORCE ON Q_2?


Q_2
$\vec{F}_2$
r
2m
θ_2
Q_1
3m
x

Q_2 IS REPELLED FROM Q_1:

$$\theta_2 = \tan^{-1}\left(\frac{2\text{ m}}{3\text{ m}}\right)$$
$$= \tan^{-1} .67 = \boxed{33.69°}$$

c) DIRECTION OF FORCE ON Q_1?


θ_1
$\vec{F}_2$
Q_1
θ_2
x
$\vec{F}_1$

IT SHOULD BE OPPOSITE TO FORCE ON Q_2, SO:

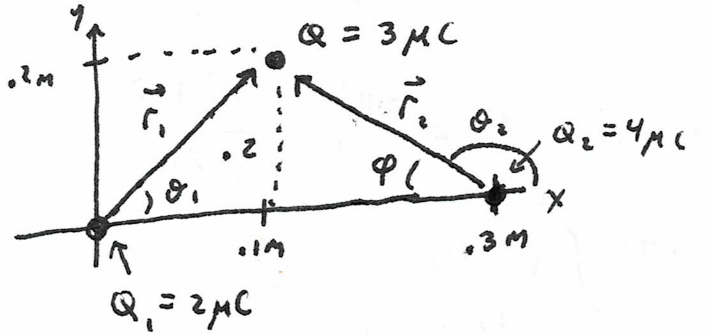
$$\theta_1 = \theta_2 + 180° = 33.69° + 180°$$
$$= \boxed{213.69°}$$

(OR EQUIVALENTLY, 33.69° BELOW NEGATIVE X-AXIS)

d) WHAT HAPPENS IF Q_2 BECOMES NEGATIVE?

- MAGNITUDES ARE UNCHANGED: $F_1 = F_2 = .00415$ N
- DIRECTIONS ARE INTERCHANGED: $\begin{cases} \theta_1 = 33.69° \\ \theta_2 = 213.69° \end{cases}$

⑤



FIND FORCE ON Q,

DISTANCES:

$$r_1 = \sqrt{(.1)^2+(.2)^2} = .224\ M$$

$$r_2 = \sqrt{(.2)^2+(.2)^2} = .283\ M$$

(OR $r_1^2 = .05\ M^2$, $r_2^2 = .08\ M^2$)

MAGNITUDES:

$$F_1 = \frac{kQQ_1}{r_1^2} = 1.08\ N \quad , \quad F_2 = \frac{kQQ_2}{r_2^2} = 1.35\ N$$

DIRECTIONS:

$$\theta_1 = TAN^{-1}\left(\frac{r_{1y}}{r_{1x}}\right) = TAN^{-1}\left(\frac{.2}{.1}\right) = 63.4^\circ$$

$$\theta_2 = 180^\circ - \varphi = 180^\circ - TAN^{-1}\left(\frac{.2}{.2}\right) = 135^\circ$$

COMPONENTS:

$$\left.\begin{aligned} F_{1x} &= F_1 \cos\theta_1 = .484\ N \\ F_{1y} &= F_1 \sin\theta_1 = .966\ N \\ F_{2x} &= F_2 \cos\theta_2 = -.955\ N \\ F_{2y} &= F_2 \sin\theta_2 = +.955\ N \end{aligned}\right\} \Rightarrow$$

COMPONENTS OF TOTAL FORCE:

$$F_x = F_{1x} + F_{2x} = .484 + (-.955) = \boxed{-.471\ N}$$

$$F_y = F_{1y} + F_{2y} = .966 + .955 = \boxed{1.921\ N}$$

MAGNITUDE & DIRECTION OF TOTAL FORCE:

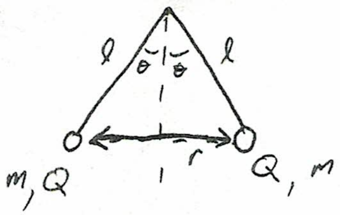
$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(-.471)^2 + (1.921)^2} = \boxed{1.98\ N}$$

$$\theta = TAN^{-1}\left(\frac{F_y}{F_x}\right) = TAN^{-1}\left(\frac{1.921}{-.471}\right) = -76.2^\circ$$

BUT THIS IS IN THE WRONG QUADRANT, SO ROTATE BY 180° TO PUT IN 2nd QUADRANT: $\theta = -76.2^\circ + 180^\circ = \boxed{103.8^\circ}$

⑥

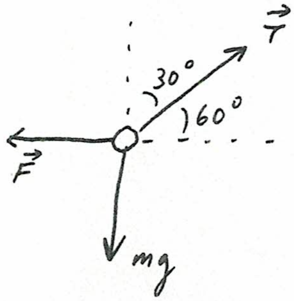
TWO CHARGED MASSES HANG ON STRINGS. IF $\ell = 1\,m$, $\theta = 30^\circ$, & $Q = 3.36 \times 10^{-5}\,C$, WHAT IS m?



FIRST, FIND r:

$$r = 2\ell \sin\theta = 2(1)(.5) = 1\,m$$

NOW, LOOK AT FORCES ON LEFT-HAND CHARGE:



$$\Sigma F_y = 0 \Rightarrow 0 = T \sin 60^\circ - mg$$

$$\therefore\ T = \frac{mg}{\sin 60^\circ}$$

$$\Sigma F_x = 0 \Longrightarrow T \cos 60^\circ - F = 0$$

$$T \cos 60^\circ = F$$

$$\left(\frac{mg}{\sin 60^\circ}\right) \cos 60^\circ = \frac{kQ^2}{r^2}$$

$$mg \cot 60^\circ = \frac{kQ^2}{r^2}$$

$$m = \frac{kQ^2}{g r^2 \cot 60^\circ} = \frac{(9 \times 10^9)(3.36 \times 10^{-5})^2}{(9.8)(1.0)^2 \cot 60^\circ} = \boxed{1.79\ kg}$$

CURRENT EXAMPLE

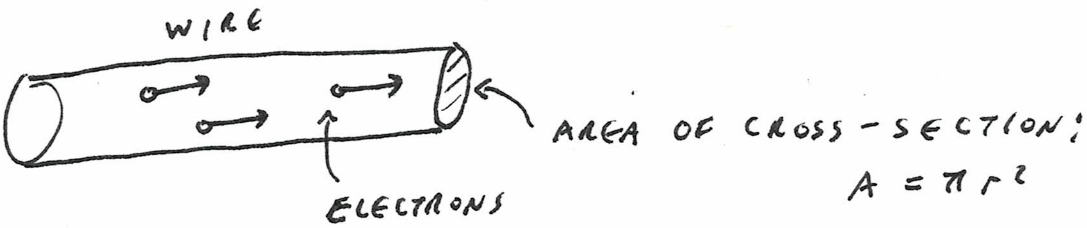
WIRE: $r = 4\,mm = .004\,m$

ELECTRON DENSITY: $n = \left(\begin{matrix}\#\text{ ELECTRONS}\\ \text{PER VOLUME}\end{matrix}\right) = 2 \times 10^{28}\ m^{-3}$

AVERAGE ELECTRON SPEED: $v = 3 \times 10^{-6}\ \frac{m}{s}$

WHAT IS THE SIZE OF THE CURRENT?

$$
\begin{aligned}
I &= \left|\text{CHARGE PER TIME}\right| = \left|\frac{dq}{dt}\right| \\
&= \left|\left(\frac{\text{CHARGE}}{\text{VOLUME}}\right)\left(\frac{\text{VOLUME}}{\text{TIME}}\right)\right| \\
&= \left|\frac{\text{CHARGE}}{\text{ELECTRON}}\right| \cdot \left(\frac{\text{ELECTRONS}}{\text{VOLUME}}\right) \cdot \left(A\,\frac{dx}{dt}\right) && \leftarrow \left(\begin{matrix}\text{ELECTRON}\\ \text{CHARGE:}\\ q = -e\end{matrix}\right) \\
&= (e)(n)(Av) \\
&= ne(\pi r^2)v && \leftarrow A = \pi r^2 \\
&= (2 \times 10^{28})(1.6 \times 10^{-19})(\pi)(.004)^2(3 \times 10^{-6}) \\
&= .4825\ \frac{C}{s} \\
&= \boxed{.4825\ A} && \left(1\ AMP = 1\ \frac{\text{COULOMB}}{s}\right)
\end{aligned}
$$



ELECTRIC FIELD = FORCE PER UNIT CHARGE: $\vec{E} = \frac{\vec{F}}{q}$

(UNITS: $\frac{N}{C}$ OR $\frac{V}{m}$)

$\vec{E}$ TELLS WHAT FORCE ON A CHARGE WOULD BE IF THE CHARGE WAS PLACED AT THAT POINT.

NOTE: THE ELECTRIC FIELD AT A POINT CAN BE NONZERO **EVEN IF** THERE IS **NO** CHARGE AT THAT POINT.

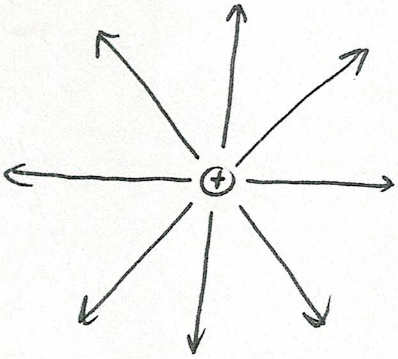
(OF COURSE, IF THERE IS NO CHARGE THEN THERE WILL BE NO FORCE THERE.)

FIELD LINES = PICTORIAL REPRESENTATION OF $\vec{E}$.

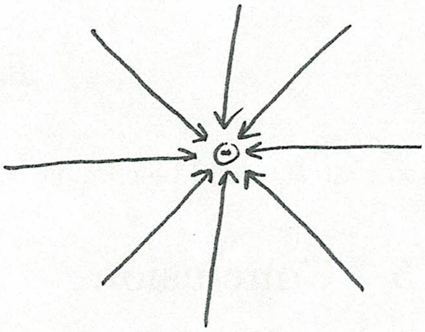
LINES POINT **AWAY** FROM **POSITIVE** CHARGES, AND **TOWARD** **NEGATIVE** CHARGES.

(THE DIRECTION OF A FIELD LINE GIVES THE DIRECTION A **POSITIVE** CHARGE WOULD MOVE IF PLACED ON THE LINE.)

EXAMPLES:

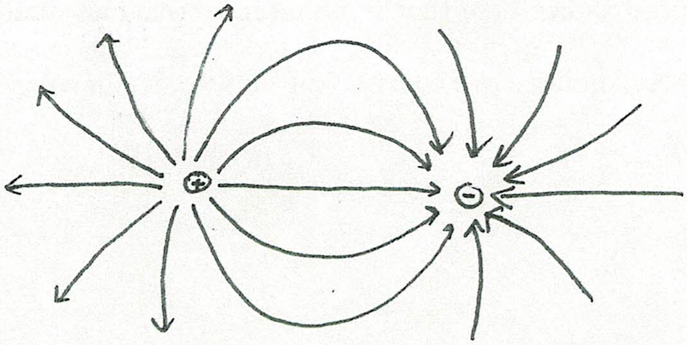


POSITIVE POINT CHARGE



NEGATIVE POINT CHARGE

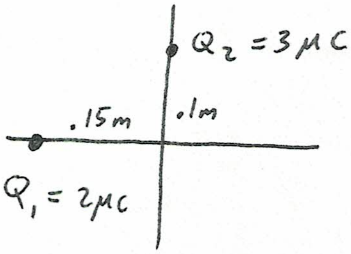
ELECTRIC DIPOLE:



POINT CHARGE:

$$E = \frac{F}{q} = \left(\frac{kqQ}{r^2}\right)\frac{1}{q} = \frac{kQ}{r^2} \rightarrow \boxed{E = \frac{kQ}{r^2}}$$

EXAMPLE:



a) WHAT IS $\vec{E}$ AT THE ORIGIN?

i) $\vec{E}$ FROM Q_1 ALONE:

$$E_{1y} = 0$$

$$E_{1x} = \frac{kQ_1}{r_1^2} = \frac{(9\times10^9)(2\times10^{-6})}{(.15)^2} = 8\times10^5 \frac{V}{m}$$

ii) $\vec{E}$ FROM Q_2 ALONE:

$$E_{2x} = 0 \quad , \quad E_{2y} = -\frac{kQ_2}{r_2^2} = -\frac{(9\times10^9)(3\times10^{-6})}{(.1)^2} = -2.7\times10^6 \frac{V}{m}$$

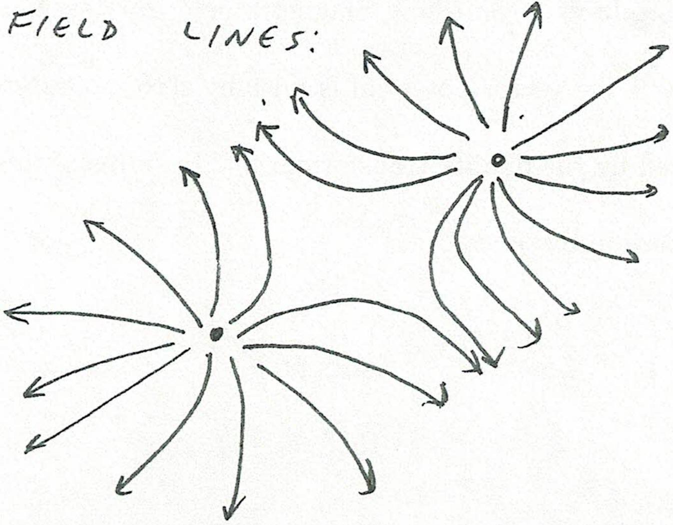
iii) TOTAL ELECTRIC FIELD:

$$E_x = E_{1x} + E_{2x} = 8\times10^5 + 0 = \boxed{8\times10^5 \frac{V}{m}}$$

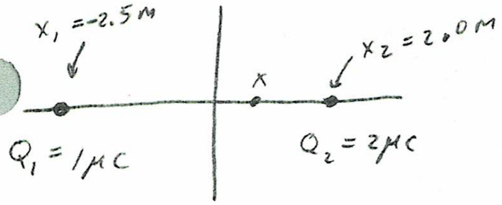
$$E_y = E_{1y} + E_{2y} = 0 + (-2.7\times10^6) = \boxed{-2.7\times10^6 \frac{V}{m}}$$

$$\left(\begin{array}{ll} \text{MAGNITUDE:} & E = \sqrt{E_x^2 + E_y^2} = 2.82\times10^6 \frac{V}{m} \\ \text{DIRECTION:} & \theta = \tan^{-1}\frac{E_y}{E_x} = -73.5^\circ \end{array}\right)$$

b) SKETCH FIELD LINES:



EXAMPLE



AT WHAT POINT WILL $\vec{E}_{TOTAL} = 0$?

IT WILL BE ON X-AXIS, AT SOME PT. X

$E_1 = \frac{kQ_1}{(x-x_1)^2}$ (TO RIGHT) $\rightarrow$ $E_{1x} = \frac{+kQ_1}{(x-x_1)^2}$

$E_2 = \frac{kQ_2}{(x-x_2)^2}$ (TO LEFT) $\rightarrow$ $E_{2x} = \frac{-kQ_2}{(x-x_2)^2}$

$$\vec{E}_{TOT} = 0 \quad \rightarrow \quad \vec{E}_1 + \vec{E}_2 = 0 \quad \xrightarrow[x\text{-COMP.}]{} \quad \frac{kQ_1}{(x-x_1)^2} - \frac{kQ_2}{(x-x_2)^2}$$

$$\frac{Q_1}{(x-x_1)^2} = \frac{Q_2}{(x-x_2)^2} \quad \rightarrow \quad Q_1(x-x_2)^2 = Q_2(x-x_1)^2$$

$$(x-x_2)^2 = 2(x-x_1)^2 \quad (\text{SINCE } Q_2 = 2Q_1)$$

$$x^2 - 2xx_2 + x_2^2 = 2x^2 - 4xx_1 + 2x_1^2$$

$$0 = x^2 + (2x_2 - 4x_1)x + (2x_1^2 - x_2^2)$$

$$0 = x^2 + [2(2) - 4(-2.5)]\,x + [2(-2.5)^2 - (2)^2]$$

$$0 = x^2 + 14x + 8.5$$

$$x = \frac{-14 \pm \sqrt{(14)^2 - 4(8.5)}}{2(1)} = \frac{-14 \pm \sqrt{162}}{2} = -7 \pm 6.36$$

$$= \boxed{-.64\,m} \quad \text{OR} \quad -13\xcancel{.36}\,m \quad \leftarrow \text{EXCLUDE}$$

EXAMPLE: AN ELECTRON PLACED IN FIELD $E = 2 \times 10^{-10} \frac{N}{C}$, POINTING TO RIGHT. FIND ITS ACCELERATION.

$$F = F_x = qE = (-e)E = -(1.6 \times 10^{-19})\, 2 \times 10^{-10} = -3.2 \times 10^{-29}\,N$$

$(F_y = 0)$

$$a_y = \frac{F_y}{m} = 0$$

$$a_x = \frac{F_x}{m} = -35\,\frac{m}{s^2}$$

$$\boxed{a = 35\,\frac{m}{s^2}, \text{ TO LEFT}}$$

CHARGE DISTRIBUTIONS

UP TILL NOW, WE'VE ONLY LOOKED AT POINT CHARGES. MAIN RESULTS:

$$F = \frac{k q_1 q_2}{r^2}$$

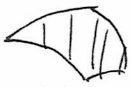
$$E = \frac{k q_1}{r^2}$$

BUT CHARGE CAN ALSO BE SMEARED OVER REGIONS OF FINITE SIZE:

1. A LINE OF CHARGE HAS A LINEAR CHARGE DENSITY:

$$\lambda = \frac{dq}{dx} = \left(\begin{array}{c}\text{CHARGE PER}\\ \text{LENGTH}\end{array}\right), \quad \text{IN } C/m$$

2. A SURFACE OR AREA HAS A SURFACE CHARGE DENSITY:



$$\sigma = \frac{dq}{dA} = \left(\begin{array}{c}\text{CHARGE PER}\\ \text{AREA}\end{array}\right), \quad \text{IN } C/m^2$$

3. A VOLUME CAN HAVE A VOLUME CHARGE DENSITY:

$$\rho = \frac{dq}{dV} = \left(\begin{array}{c}\text{CHARGE PER}\\ \text{VOLUME}\end{array}\right), \quad \text{IN } C/m^3$$

HOW CAN YOU FIND THE FIELD FROM A CHARGE DISTRIBUTION? TWO METHODS: 1. GAUSS' LAW (NEXT CHAPTER)

2. INTEGRATION

INTEGRATION METHOD!

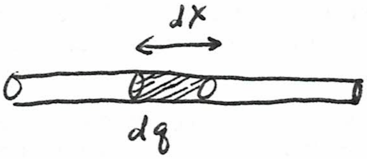
1. BREAK INTO LOTS OF POINT CHARGES $dq = \begin{cases} \lambda\, dx \\ \sigma\, dA \\ \rho\, dV \end{cases}$.

2. FIND EXPRESSION FOR FIELD FROM EACH POINT CHARGE.

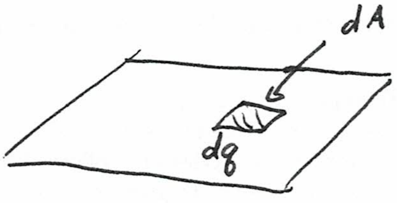
3. ADD UP THE EFFECT OF ALL THE POINT CHARGES BY INTEGRATING.

EXAMPLE ON NEXT PAGE

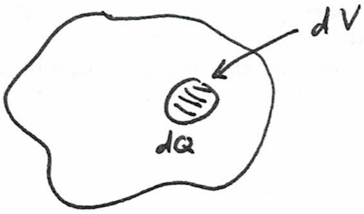
LINE OF CHARGE:



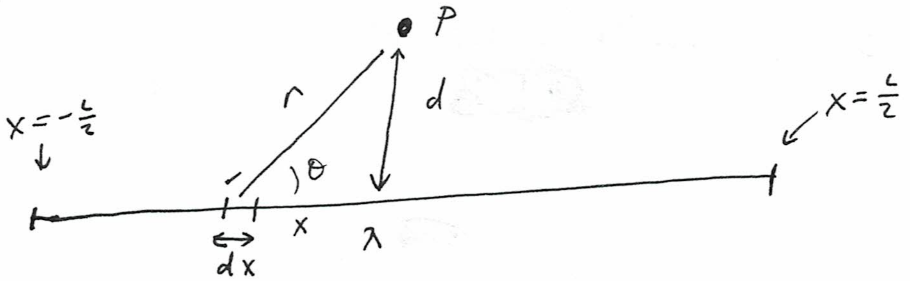
AREA CHARGE:



VOLUME CHARGE:



EXAMPLE:



FIND AN EXPRESSION FOR $\vec{E}$ AT DISTANCE d FROM LENGTH-L LINE OF CHARGE DENSITY λ C/M.

FIRST, BY SYMMETRY $E_x = 0$. ONLY NEED E_y.

DISTANCE: $r = \sqrt{d^2+x^2}$

MAGNITUDE: $dE = \frac{k\,dq}{r^2} = \frac{k\lambda\,dx}{d^2+x^2}$

y-COMPONENT: $dE_y = dE \sin\theta = \underbrace{\left(\frac{k\lambda\,dx}{d^2+x^2}\right)}_{dE} \underbrace{\frac{d}{\sqrt{d^2+x^2}}}_{\sin\theta}$

SO:

$$E_y = \int dE_y = \int_{-L/2}^{L/2} \frac{dk\lambda\,dx}{(d^2+x^2)^{3/2}} = dk\lambda \int_{-L/2}^{L/2} \frac{dx}{(d^2+x^2)^{3/2}}$$

$$= dk\lambda \left(\frac{x}{d^2(x^2+d^2)^{1/2}} \right) \Bigg|_{-\frac{L}{2}}^{L/2}$$ (#19, APPENDIX E: $\overset{\text{PAGE \#}}{\overbrace{A-11}}$) (OR, INSIDE FRONT COVER)

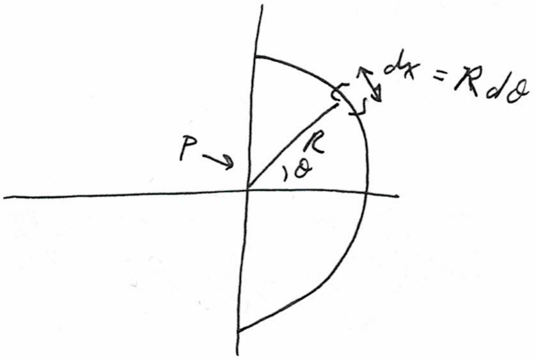
$$= \frac{k\lambda}{d} \left[\frac{L/2}{\sqrt{d^2+\left(\frac{L}{2}\right)^2}} - \frac{(-L/2)}{\sqrt{d^2+\left(\frac{L}{2}\right)^2}} \right]$$

$$= \frac{k\lambda L}{d\sqrt{d^2+\left(\frac{L}{2}\right)^2}}$$

$$\boxed{E = \frac{k\lambda L}{d\sqrt{d^2+\left(\frac{L}{2}\right)^2}}, \quad \text{UPWARD } (y\text{-DIRECTION})}$$

NOTE! $\lim\limits_{L\to\infty} E = \frac{2k\lambda}{d} = \frac{\lambda}{2\pi\varepsilon_0 d}$

EXAMPLE



SEMICIRCLE OF RADIUS R HAS UNIFORM $\lambda = \frac{\text{CHARGE}}{\text{LENGTH}}$.

FIND FIELD AT POINT P.

BY SYMMETRY: $\boxed{E_y = 0}$

$$dq = \lambda\, dx = \lambda R\, d\theta$$

$$dE = \frac{k\,dq}{R^2} = \frac{k\lambda R\, d\theta}{R^2} = \frac{k\lambda}{R}\, d\theta$$

$$dE_x = -dE \cdot \cos\theta = -\left(\frac{k\lambda}{R}\, d\theta\right)\cos\theta$$

$$E_x = -\int_{-\pi/2}^{\pi/2} \frac{k\lambda}{R}\cos\theta\, d\theta = -\frac{k\lambda}{R}\sin\theta\Big|_{-\pi/2}^{\pi/2} = -\frac{2k\lambda}{R}$$

$$\boxed{E_x = \frac{-2k\lambda}{R}, \quad E_y = 0}$$

CONDUCTORS AND ELECTRIC FIELDS

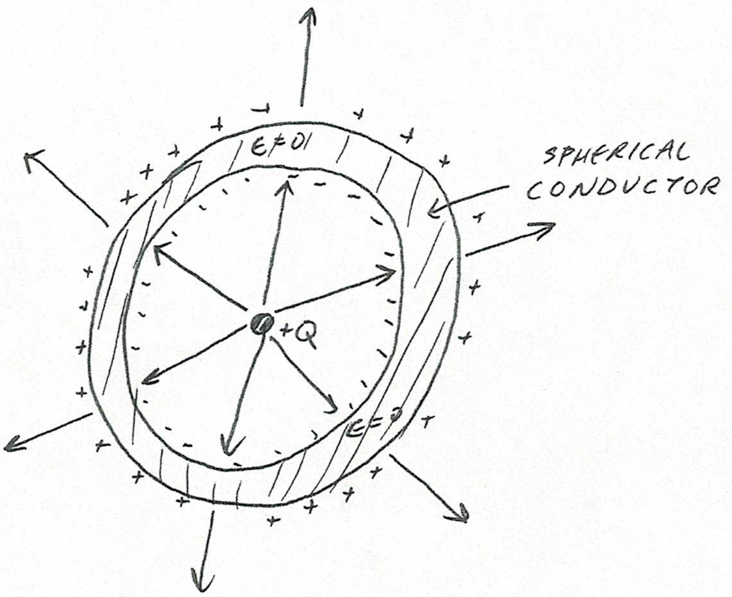
IN A CONDUCTOR CHARGES ARE FREE TO MOVE AROUND. THEY WILL NATURALLY REDISTRIBUTE THEMSELVES TO MINIMIZE THE ENERGY. THIS LEADS TO THE FOLLOWING:

1. THE CHARGES WANT TO BE AS FAR APART AS POSSIBLE.
   ∴ $\boxed{\text{ALL CHARGE WILL RESIDE ON SURFACE.}}$

2. THE ELECTRIC FIELD DUE TO CHARGES IN CONDUCTOR WILL CANCEL OUT ANY APPLIED FIELD.
   $\boxed{E = 0 \text{ IN CONDUCTOR}}$ (STATIC CASE)

3. $\boxed{\text{AT SURFACE, } \vec{E} \text{ IS } \perp \text{ TO CONDUCTOR}}$
   OTHERWISE THERE WOULD BE FORCE $\vec{F} = q\vec{E}$ MOVING CHARGES. THE CHARGES WOULD MOVE AROUND UNTIL $\vec{F}$ VANISHES.

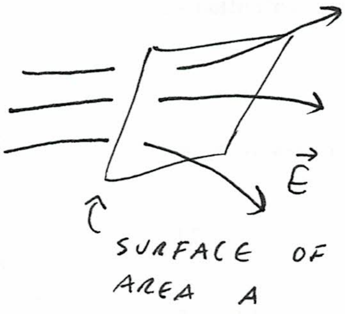
EXAMPLE: POINT CHARGE IN A CONDUCTING SHELL.



INDUCED CHARGES:
INNER SURFACE ⇒ $-Q$
OUTER SURFACE ⇒ $+Q$

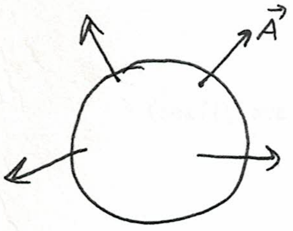
CH. 22 - GAUSS' LAW

- GAUSS' LAW GIVES ANOTHER WAY OF COMPUTING $\vec{E}$.
- IT IS MORE GEOMETRICAL & TAKES ADVANTAGE OF SYMMETRIES.
- IN SYSTEMS WITH HIGH DEGREE OF SYMMETRY, IT MAKES CALCULATIONS MUCH EASIER.

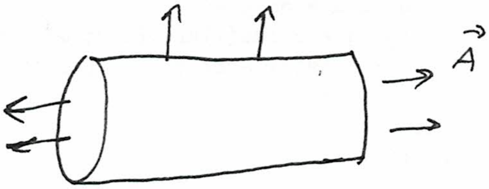


ELECTRIC FLUX: Φ MEASURE AMOUNT OF FIELD PASSING THROUGH SURFACE.

FIRST, DEFINE AREA VECTOR OF SURFACE; $\vec{A}$ HAS MAGNITUDE EQUAL TO AREA OF SURFACE. DIRECTION IS ⊥ TO SURFACE, POINTING OUTWARD (FOR CLOSED SURFACE).



SPHERE



CYLINDER

ELECTRIC FLUX IS DEFINED BY:

$$\boxed{\Phi = \int \vec{E} \cdot d\vec{A}}$$

UNITS: $V \cdot m = \frac{Nm}{C}$

V = VOLTS (DEFINED IN CH. 23)

DOT PRODUCT PROJECTS OUT PART OF $\vec{E}$ ⊥ TO SURFACE.

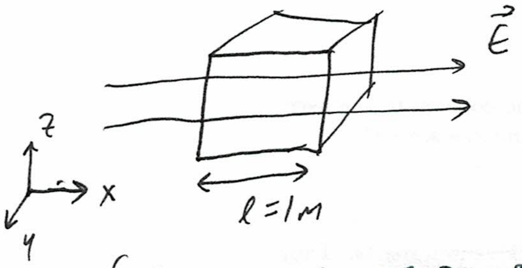
IF SURFACE CAN BE BROKEN INTO SEVERAL REGIONS, WITH $\vec{E}$ = CONSTANT ON EACH REGION, THEN:

$$\Phi = \sum_i \vec{E}_i \cdot \vec{A}_i$$

WHERE i LABELS THE CONSTANT-$\vec{E}$ REGIONS.

EXAMPLES

1. UNIFORM FIELD $\vec{E} = 5 \frac{V}{m} \hat{i}$ PASSING THROUGH CUBE OF SIDE LENGTH $l = 1m$, AS SHOWN.



AREA OF EACH SIDE IS $A = l^2 = 1\ m^2$.

FOR THE TOP & BOTTOM SIDES: $\vec{A} = \pm(1m^2)\hat{k}$

FOR THE TWO SIDES IN X-Z PLANE: $\vec{A} = \pm(1m^2)\hat{j}$

FOR THESE 4 SIDES, $\vec{E} \cdot \vec{A}$ DOESN'T CONTRIBUTE TO FLUX.

FOR THE REMAINING ENDS, THE LEFT & RIGHT ENDS HAVE: $\vec{A}_L = -1m^2\,\hat{i}$, $\vec{A}_R = +1m^2\,\hat{i}$.

SO FLUX IS:

$$\Phi = \sum_{\text{SIDES}} \vec{E} \cdot \vec{A} = 0 + 0 + 0 + 0 + (-1m^2) \cdot E + (+1m^2)E$$

$$= \boxed{0} \Rightarrow \text{NO NET FLUX.}$$

2. a) FIND FLUX DUE TO PROTON THROUGH SPHERES OF RADIUS 1M & 2M, CENTERED ON PROTON.

(i) AT 1m:

$$E = \frac{ke}{r^2} = \frac{(9\times10^{9})(1.6\times10^{-19})}{(1m)^2} = 1{,}44\times10^{-9}\ \frac{V}{m}\ \text{(OUTWARD)}$$

$\vec{A}$: MAGNITUDE = $A = 4\pi r^2 = 12{,}57\ m^2$ (OUTWARD)

E IS CONSTANT ON THE SPHERE, SO:

$$\Phi = E\cdot A = (1.44\times10^{-9})(12{,}57) = \boxed{1{,}8\times10^{-8}\ Vm}$$

(ii) AT 2m:

NOW: $$E = \frac{(9\times10^{9})(1.6\times10^{-19})}{(2m)^2} = 3{,}6\times10^{-10}\ \frac{V}{m}\ \text{(OUTWARD)}$$

$$A = 4\pi r^2 = 50.3\ m^2\ \text{(OUTWARD)}$$

SO: $$\Phi = (3.6\times10^{-10})(50{,}3) = \boxed{1{,}8\times10^{-8}\ V.m}$$

IT IS NOT A COINCIDENCE THAT THESE ARE THE SAME, AS WE'LL SEE BELOW.

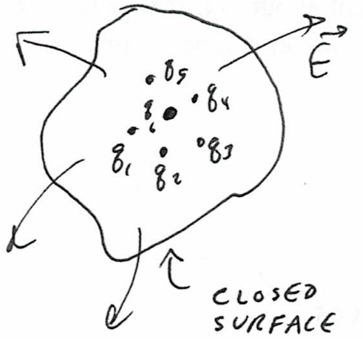
b) WHAT IS FLUX AT 1 m IF PROTON IS REPLACED BY AN ELECTRON.

ONLY CHANGE IS THAT $\vec{E}$ REVERSES DIRECTION, POINTING INWARD NOW.

SO NOW $\vec{E}$ & $\vec{A}$ ARE 180° FROM EACH OTHER, INTRODUCING A MINUS SIGN.

SO: $$\boxed{\Phi = \underset{\underset{\text{OUTWARD}}{\uparrow}}{-}1{,}8\times10^{-8}\ V.m}$$

GAUSS' LAW RELATES FLUX THROUGH CLOSED SURFACE TO CHARGE ENCLOSED BY THE SURFACE:

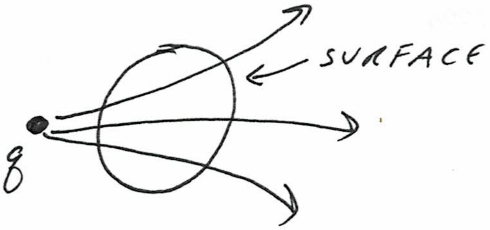


$$\boxed{\varepsilon_0 \int E \cdot dA = q_{enc}}$$

FLUX, Φ

CHARGE ENCLOSED BY SURFACE

- NOTE THAT BENDING OR STRETCHING SURFACE DOES NOT AFFECT FLUX, AS LONG AS SURFACE DOESN'T CROSS A CHARGE. (THIS EXPLAINS LAST EXAMPLE: CHANGING RADIUS OF SPHERE DOESN'T CHANGE FLUX THROUGH IT.)
- POSITIVE CHARGES ENCLOSED: $\Phi > 0$
- NEGATIVE CHARGE ENCLOSED: $\Phi < 0$
- CHARGES OUTSIDE SURFACE DON'T CONTRIBUTE TO FLUX:



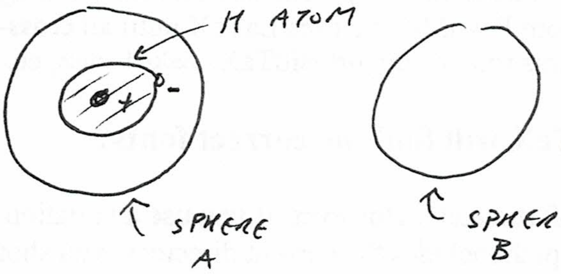
EACH FLUX LINE ENTERS ONCE & LEAVES ONCE, GIVING A POSITIVE & NEGATIVE CONTRIBUTION THAT CANCEL.

STRATEGY FOR USING GAUSS TO FIND $\vec{E}$ FIELDS.

- DRAW AN IMAGINARY SURFACE (A GAUSSIAN SURFACE) AROUND CHARGES.
- CHOOSE SURFACE TO HAVE SAME SYMMETRY AS CHARGES $\Rightarrow$ SURFACE BREAKS INTO REGIONS OF CONSTANT E $\Rightarrow$ AVOIDS DOING INTEGRALS.
- COMPUTE Φ & PLUG INTO GAUSS TO FIND $\vec{E}$.

EXAMPLES

1. FIND FLUX FROM HYDROGEN ATOM THROUGH THE TWO SPHERES SHOWN!



(THE ATOM IS INSIDE SPHERE A, OUTSIDE SPHERE B.)

A: TOTAL CHARGE ENCLOSED: $q_{enc} = (+e) + (-e) = 0$

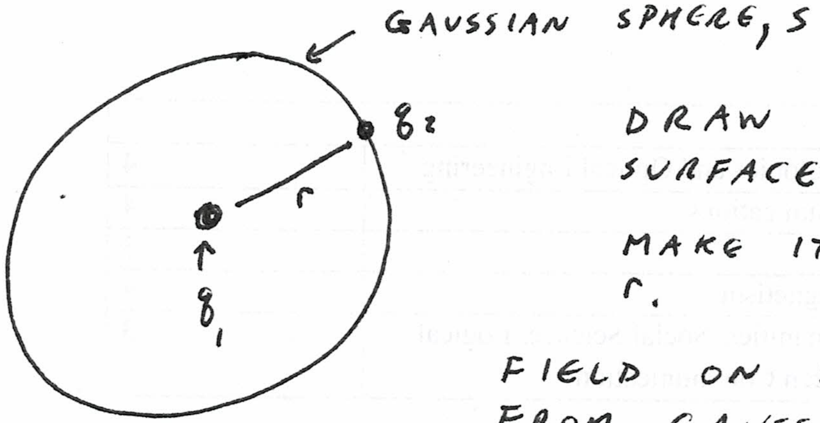
B: TOTAL CHARGE ENCLOSED: $q_{enc} = 0$

IN BOTH CASES: $\boxed{\Phi = 0}$ BY GAUSS

2. DERIVE COULOMB'S LAW FROM GAUSS' LAW.

(NEXT PAGE) →

2. DERIVE COULOMB'S LAW FROM GAUSS' LAW.



DRAW IMAGINARY GAUSSIAN SURFACE CENTERED ON q_1. MAKE IT SPHERICAL OF RADIUS r.

FIELD ON SPHERE CAN BE FOUND FROM GAUSS:

$$\int_S \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \rightarrow E(4\pi r^2) = \frac{q_1}{\varepsilon_0}$$

$$\downarrow$$

$$E = \frac{q_1}{4\pi\varepsilon_0 r^2}$$

SO FORCE ON q_2 PLACED AT DISTANCE r FROM q_1 IS:

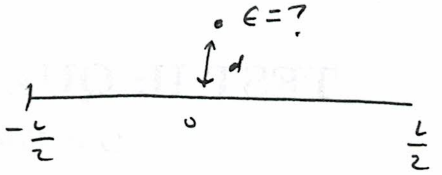
$$F = q_2 E = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} = \boxed{\frac{k q_1 q_2}{r^2}}$$

$$\left(\text{SINCE } k = \frac{1}{4\pi\varepsilon_0}\right)$$

EXAMPLE

RECALL: LINE SEGMENT W/ CHARGE DENSITY λ!

$$E = \frac{k\lambda L}{d\sqrt{d^2+\left(\frac{L}{2}\right)^2}}$$

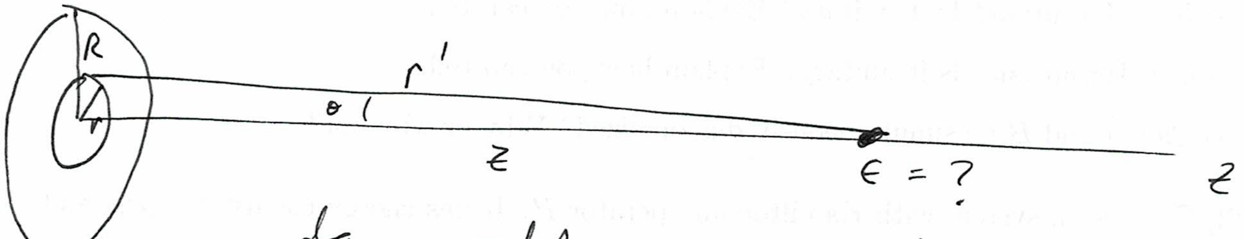


NOTE: AS $L \to \infty$:

$$E = \frac{k\lambda L}{d\sqrt{d^2+\left(\frac{L}{2}\right)^2}} \to \frac{k\lambda L}{d\sqrt{\left(\frac{L}{2}\right)^2}} = \boxed{\frac{2k\lambda}{d}} \quad \text{FOR } L\to\infty$$

$$= \frac{\lambda}{2\pi\varepsilon_0 d}$$

CHARGED PLANE: DIRECT INTEGRATION



$$dq = \sigma\, dA = \sigma\, 2\pi r\, dr$$

$$dE = \frac{k\,dq}{r'^2} = \frac{2\pi\sigma k r\, dr}{r^2+z^2}$$

$$dE_z = \frac{2\pi\sigma k r\,dr}{r^2+z^2}\cos\theta = \frac{2\pi\sigma k r\,dr}{r^2+z^2}\,\frac{z}{\sqrt{r^2+z^2}}$$

$$E_z = \int_0^R 2\pi\sigma k z \frac{r\,dr}{(r^2+z^2)^{3/2}} \overset{y=r^2}{=} 2\pi\sigma k z\left(\frac{1}{2}\right)\int_0^{R^2}\frac{dy}{(y+z^2)^{3/2}}$$

$$= \pi\sigma k z\int_{z^2}^{R^2+z^2}\frac{d\lambda}{\lambda^{3/2}} \qquad (\lambda = y+z^2)$$

$$= \pi\sigma k\,(-2)\,\lambda^{-1/2}\Big|_{z^2}^{R^2+z^2}$$

$$= -\frac{1}{2\pi\sigma k z}\left(\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{z}\right) = 2\pi\sigma k z\left(\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right)$$

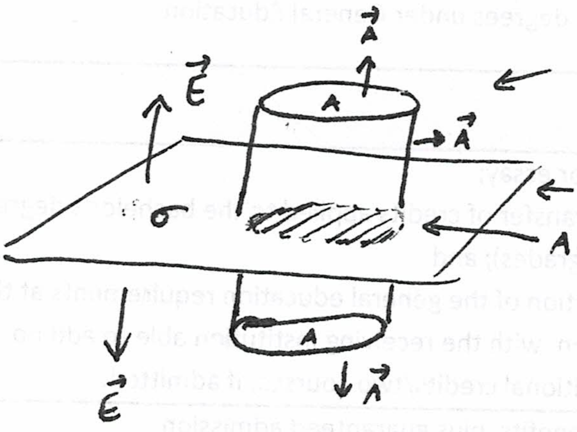
FOR $R \to \infty$:

$$E_z = 2\pi\sigma k z\left[\frac{1}{z} - \frac{1}{R}\left(1 - \frac{1}{2}\left(\frac{z}{R}\right)^2 + \dots\right)\right]$$

$$= 2\pi\sigma k = \frac{\sigma}{2\varepsilon_0} = \text{CONSTANT}$$

NOW TRY AGAIN USING GAUSS →

REDO FIELD FROM CHARGED PLANE USING GAUSS' LAW:



TAKE GAUSSIAN SURFACE TO BE CYLINDER OF AREA A AT EACH END, LENGTH L.

← CHARGED PLANE

← AREA = A → $q_{enc} = \sigma A$

← INTEGRAL OVER CYLINDER

$$\oint \vec{E}\cdot d\vec{A} = \underset{\text{SIDE}}{\int \vec{E}\cdot d\vec{A}} + \underset{\text{TOP}}{\int \vec{E}\cdot d\vec{A}} + \underset{\text{BOTTOM}}{\int \vec{E}\cdot d\vec{A}}$$

$$= \underset{\substack{\uparrow \\ \vec{E}\perp d\vec{A} \text{ ON SIDE (SEE PICTURE ABOVE)}}}{0} + \underbrace{EA + EA}_{\vec{E} \text{ IS CONSTANT \& PARALLEL TO } \vec{A} \text{ ON TOP \& BOTTOM OF CYLINDER}}$$

$$= 2EA$$

GAUSS:

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \rightarrow 2EA = \frac{\sigma A}{\varepsilon_0} \rightarrow 2E = \frac{\sigma}{\varepsilon_0}$$

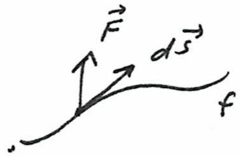
$$\downarrow$$

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}$$

(SAME RESULT AS BY OTHER METHOD)

CH. 23 - ELECTRIC POTENTIAL

RECALL POTENTIAL ENERGY & WORK:



$$\Delta U = -W = -\int \vec{F} \cdot d\vec{s}$$

WE CAN DEFINE AN ELECTROSTATIC POTENTIAL ENERGY DUE TO THE ELECTROSTATIC FORCE.

ONLY DIFFERENCES MATTER: $\Delta U = U_f - U_i$.

BUT IT IS STANDARD TO TAKE INITIAL POINT AT ∞ & DEFINE $U_i = 0$. SO THEN U CAN BE DEFINED AT INDIVIDUAL POINTS:

$U = -W_\infty$, WHERE W_∞ = WORK TO BRING CHARGE IN FROM ∞.

FOR POINT CHARGES, INTEGRATING COULOMB'S LAW GIVES:

$$\boxed{U = \frac{kq_1q_2}{r}}$$ POINT CHARGES

UNITS: JOULES

- ALTERNATE UNIT:

ELECTRON-VOLT: $1 \text{ eV} = 1.6 \times 10^{-19}$ J

- NOTE AGAIN: U = SCALAR, NOT VECTOR.

VOLTAGE

- JUST AS WE DEFINED THE ELECTRIC FIELD OF A CHARGE BY DIVIDING F/q, WE CAN DIVIDE U/q TO DEFINE A NEW QUANTITY:

$$\boxed{\underline{\text{POTENTIAL}} \text{ OR } \underline{\text{VOLTAGE}}: \quad V = U/q}$$

- UNITS OF VOLTAGE: $J/C \equiv$ **VOLTS** (V)
- ONLY DIFFERENCES IN POTENTIAL MATTER. V & ΔV ARE USED INTERCHANGEABLY AND MAY DENOTE DIFFERENCE BETWEEN FINITE POINTS i & f, OR BETWEEN POINT f & ∞.
- V IS SCALAR
- NOW CAN SEE SIGNIFICANCE OF DEFINITION OF eV:

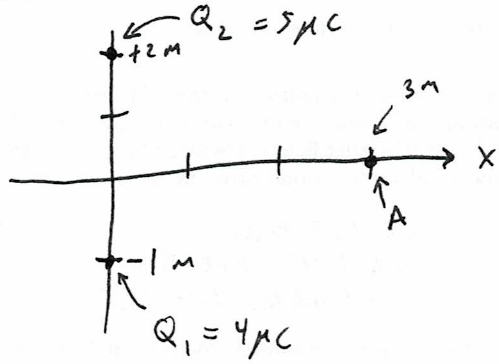
1 eV = ENERGY GAINED BY ELECTRON WHEN ACCELERATED THROUGH VOLTAGE DIFFERENCE OF 1 V.

- NOTE: V LARGEST WHEN $\mathcal{E}$ LARGE, SO

$$\left.\begin{array}{l} +q \rightarrow \text{WANTS TO MOVE TOWARD LOWER } V \\ -q \rightarrow \text{WANTS TO MOVE TOWARD HIGHER } V \end{array}\right\} \text{TO MINIMIZE POTENTIAL ENERGY}$$

EXAMPLES WITH POINT CHARGES

①



a) FIND $\vec{E}$ AT POINT A.

$$E_1 = \frac{kq_1}{r_1^2} = (9\times10^9)\left(\frac{4\times10^{-6}}{3^2+1^2}\right) = 3600\ \frac{V}{m}$$

$$\theta_1 = \tan^{-1}\frac{1}{3} = 18.4^\circ$$

$$E_2 = \frac{kq_2}{r_2^2} = (9\times10^9)\left(\frac{5\times10^{-6}}{3^2+2^2}\right) = 3461.5\ \frac{V}{m}$$

$$\theta_2 = -\tan^{-1}\frac{2}{3} = -33.7^\circ$$

TOTAL: $E_x = E_{1x} + E_{2x} = E_1\cos\theta_1 + E_2\cos\theta_2 = \boxed{6295.76\ \frac{V}{m}}$

$E_y = E_1\sin\theta_1 + E_2\sin\theta_2 = \boxed{-784.26\ N/C}$

$E = \sqrt{E_x^2+E_y^2} = \boxed{6344.42\ \frac{V}{m}}$ $\qquad \theta = \tan^{-1}\frac{E_y}{E_x} = \boxed{-7.1^\circ}$

b) FIND POTENTIAL AT POINT A.

$$V = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} = k\left(\frac{q_1}{r_1}+\frac{q_2}{r_2}\right) = (9\times10^9)\left(\frac{4\times10^{-6}}{\sqrt{10}} + \frac{5\times10^{-6}}{\sqrt{13}}\right)$$

$$= \boxed{23{,}865\ V}$$

c) SUPPOSE A CHARGE $q = 3\mu C$ IS PLACED AT POINT A, WHAT IS ITS POTENTIAL ENERGY?

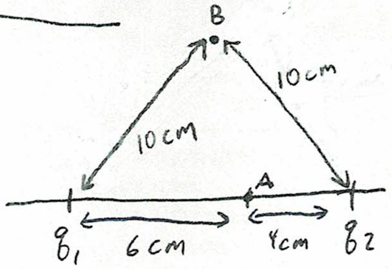
$$U = \frac{kqq_1}{r_1} + \frac{kqq_2}{r_2} = kq\left(\frac{q_1}{r_1}+\frac{q_2}{r_2}\right) = \boxed{.071\ J}$$

– OR, MORE SIMPLY –

$$U = Vq = (3\times10^{-6}c)(23865\ V) = \boxed{.071\ J}$$

EXAMPLES

② $q_1 = +12 \times 10^{-9}\,C$ $\quad q_2 = -q_1$



a) FIND POTENTIAL AT POINT B.

FROM 1ST CHARGE:

$$V_1 = \frac{kq_1}{r_1} = \frac{(9\times10^9)(12\times10^{-9})}{.1} = 1080\ V$$

2nd CHARGE:

$$V_2 = -1080\ V$$

TOTAL: $V(B) = V_1 + V_2 = \boxed{0}$

b) FIND V AT POINT A.

$V_1 = 1800\ V$ $\qquad V_2 = -2700\ V$

$$V(A) = 1800 - 2700 = \boxed{-900\ V}$$

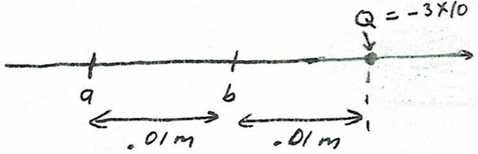
c) HOW MUCH WORK WOULD IT TAKE TO MOVE AN ELECTRON TO POINT A FROM INFINITY?

$$\Delta PE = Q_{ELECTRON}\,\Delta V = (-e)\big(V(A) - V(\infty)\big)$$

$$= (-1.6\times10^{-19}\,C)(-900 - 0) = +\boxed{1.44\times10^{-16}\ J}$$

(PROTON: $\Delta PE = -1.44\times10^{-16}\ J$)

③ A PARTICLE WITH $q = 2\times10^{-9}\,C$ & $m = .005\ kg$ STARTS AT REST AT POINT a. WHAT IS ITS SPEED AT POINT b?



i) POINT a: $K_a = 0$

$$(PE)_a = qV = \frac{-kqQ}{r} = \frac{(9\times10^9)(2\times10^{-9})(-3\times10^{-9})}{(.02)} = -2.7\times10^{-6}\ J$$

ii) POINT b: $K_b = \frac{1}{2}mv^2$

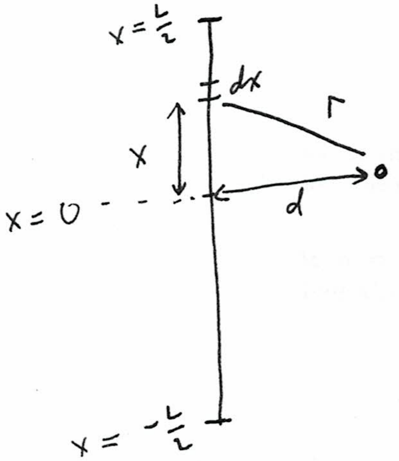
$$(PE)_b = -5.4\times10^{-6}\ J$$

SO, $(PE)_a + (KE)_a = (PE)_b + (KE)_b \Rightarrow -2.70\times10^{-6} + 0 = \frac{1}{2}mv^2 - 5.4\times10^{-6}$

$$2.7\times10^{-6} = \frac{1}{2}mv^2$$

$$v = \sqrt{\frac{2(2.7\times10^{-6})}{m}} = .033\ \frac{m}{s} = \boxed{3.3\ \frac{cm}{s}}$$

③ FIND V AT DISTANCE d FROM CENTER OF LINE OF LINEAR CHARGE DENSITY λ AND LENGTH L.



$$r = \sqrt{x^2+d^2}$$

$$dq = \lambda\, dx$$

$$V = \int_{-L/2}^{L/2} \frac{k\, dq}{r} = \int_{-L/2}^{L/2} \frac{k \lambda\, dx}{\sqrt{x^2+d^2}}$$

$$= k\lambda \ln\left(x + \sqrt{x^2+a^2}\right)\Big|_{-L/2}^{L/2}$$

$$= k\lambda \ln\left(\frac{L/2 + \sqrt{(L/2)^2+a^2}}{-L/2 + \sqrt{(L/2)^2+a^2}}\right)$$

$$\boxed{V = k\lambda \ln\left(\frac{L/2 + \sqrt{(L/2)^2+a^2}}{-L/2 + \sqrt{(L/2)^2+a^2}}\right)}$$

SUMMARY OF FIELDS & POTENTIALS

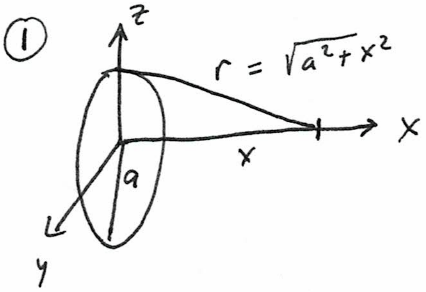
GENERAL	POINT CHARGES	
$\vec{F} = q\vec{E}$	$F = \frac{kq_1q_2}{r^2}$	$E = \frac{kq_1}{r^2}$
$U = qV$	$U = \frac{kq_1q_2}{r}$	$V = \frac{kq_1}{r}$

FOR EXTENDED CHARGED OBJECT: TO FIND V, BREAK INTO POINT CHARGES, THEN INTEGRATE, AS WE DID FOR $\vec{E}$.

- PROCEDURE IS SIMPLER THAN FOR $\vec{E}$ SINCE V IS SCALAR: NO COMPONENTS TO WORRY ABOUT.

EXAMPLES

①



A RING HAS TOTAL CHARGE q & CHARGE PER LENGTH:

$$\lambda = \frac{q}{2\pi a}$$

WHAT IS V AT DISTANCE x FROM RING'S CENTER?

ALL POINTS ARE SAME DISTANCE FROM POINT:

$$r = \sqrt{a^2+x^2}$$

$$dq = \lambda\, ds = \lambda a\, d\theta \qquad \leftarrow (s = \text{ARC LENGTH})$$

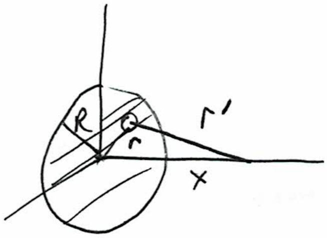
SO:

$$V = \int \frac{dq}{4\pi\varepsilon_0 r} = \int_0^{2\pi} \frac{\lambda a\, d\theta}{4\pi\varepsilon_0\sqrt{a^2+x^2}} = \frac{\lambda a}{4\pi\varepsilon_0\sqrt{a^2+x^2}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi}$$

$$\boxed{V = \frac{\lambda a}{2\varepsilon_0\sqrt{a^2+x^2}}} \qquad \left(= \frac{2\pi k \lambda a}{\sqrt{a^2+x^2}} \right)$$

NOTE: SINCE $q = 2\pi a \lambda$, THIS IS JUST $V = \frac{kq}{r}$.

② WHAT IS POTENTIAL DISTANCE X FROM AXIS OF UNIFORMLY CHARGED DISK ?



GIVEN: $\sigma = \frac{q}{A}$

R = RADIUS

NOW: $dA = r\,dr\,d\theta$

$dq = \sigma\,dA = \sigma\,r\,dr\,d\theta$

$r' = \sqrt{r^2 + x^2}$

SO:

$$V = \int \frac{k\,dq}{r} = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\,dA}{r} = \frac{\sigma}{4\pi\varepsilon_0}\int \frac{r\,dr\,d\theta}{r}$$

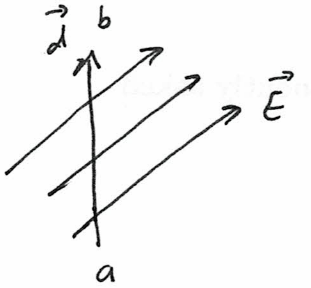
$$= \frac{\sigma}{4\pi\varepsilon_0}\int_0^R dr \int_0^{2\pi} d\theta = \frac{\sigma}{4\pi\varepsilon_0}\cdot R \cdot 2\pi$$

$$\boxed{V = \frac{\sigma R}{2\varepsilon_0}}$$

HOW ARE $\vec{E}$ & V RELATED?

$$\boxed{\begin{gathered} V = -\int E_x\,dx - \int E_y\,dy - \int E_z\cdot dz \equiv -\int \vec{E}\cdot d\vec{s} \\ \text{- OR -} \\ E_x = -\frac{\partial V}{\partial x},\quad E_y = -\frac{\partial V}{\partial y},\quad E_z = -\frac{\partial V}{\partial z} \end{gathered}}$$

SPECIAL CASE: UNIFORM $\vec{E}$



$$\Rightarrow \boxed{V_{ba} = -\vec{E}\cdot\vec{d}}$$

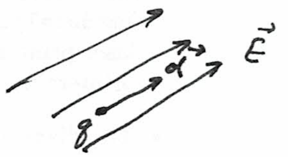
VECTOR FROM PT. a TO PT b

VOLTAGE DIFF. BETWEEN POINT a & POINT b

EXAMPLES

1. A CHARGE $q = 5\mu C$ IS DROPPED INTO A UNIFORM FIELD $E = 10$ V. HOW MUCH KINETIC ENERGY HAS IT GAINED AFTER TRAVELLING 1m?

q WILL TRAVEL ∥ TO $\vec{E}$,



SO:

$$V = -\vec{E}\cdot\vec{d} = -Ed$$

$$= -\left(10\tfrac{V}{m}\right)(1m) = -10\text{ V}$$

$$\Delta U = qV = (5\times10^{-6}\text{c})(-10\text{ V}) = -5\times10^{-5}\text{ J}$$

SO: $$\Delta K = -\Delta U = \boxed{+5\times10^{-5}\text{ J}}$$

- WHAT WOULD ANSWER BE FOR ELECTRON?

2. AN ELECTRIC POTENTIAL IS GIVEN AS A FUNCTION OF POSITION IN X-Y PLANE:

$$V(x,y) = axy^2 + bx + c\ln y$$

$(a, b, c = \text{CONSTANTS})$

WHAT IS ELECTRIC FIELD IN THIS PLANE?

$$\boxed{\begin{aligned} E_x &= -\frac{\partial V}{\partial x} = -ay^2 + b \\ E_y &= -\frac{\partial V}{\partial y} = 2axy + \frac{c}{y} \end{aligned}}$$

- OR -

$$\begin{cases} E = \sqrt{E_x^2 + E_y^2} = \sqrt{(b - ay^2)^2 + \left(2axy + \frac{c}{y}\right)^2} \\ \theta = \tan^{-1} \dfrac{2axy + \frac{c}{y}}{b - ay^2} \end{cases}$$

3. AN ELECTRIC FIELD GIVEN:

$$E_x = 5x^2 - 3x$$

WHAT IS THE POTENTIAL DIFFERENCE BETWEEN POINTS $x = 0$ & $x = 1\,m$?

$$V = -\int_{x=0}^{1} E_x\,dx = -\int_0^1 (\overset{\text{IN } \frac{V}{M^3}}{5}x^2 - \overset{\text{IN } \frac{V}{M^2}}{3}x)\,dx$$

$$= -\left(\frac{5}{3}x^3 - \frac{3}{2}x^2\right)\Big|_0^1 = -\left(\frac{5}{3} - \frac{3}{2}\right) = -\frac{10-9}{6}$$

$$= \boxed{-\frac{1}{6}\,V}$$

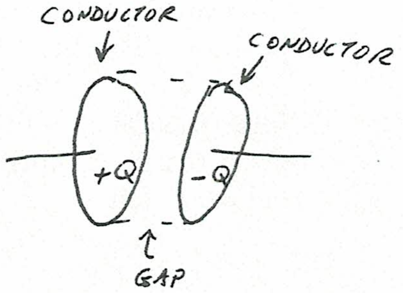
CH. 24 - CAPACITANCE

CAPACITOR = CONDENSER = PAIR OF CONDUCTORS SEPERATED BY A GAP. (GAP MAY CONTAIN AIR OR DIELECTRIC)

CAPACITORS ARE USED TO STORE ENERGY IN AN ELECTRIC FIELD. THEY ARE USEFUL BECAUSE THE ENERGY CAN BE DISCHARGED VERY RAPIDLY (AS OPPOSSED TO BATTERIES, WHICH ARE MUCH SLOWER AT TRANSFERRING ENERGY).

BASIC EQUATION:

$$\boxed{Q = CV}$$ (V = VOLTAGE DIFFERENCE)



C = THE CAPACITANCE ⇒ DEPENDS ONLY ON THE GEOMETRY OF THE CONDUCTORS & THE MATERIAL IN THE GAP.

UNITS OF CAPACITANCE: $1 \text{ FARAD (F)} = 1 \frac{C}{V} = 1 \frac{C^2}{NM}$

MORE COMMON UNITS: 1 PICOFARAD $(pF) = 10^{-12} F$

1 MICROFARAD $(\mu F) = 10^{-6} F$

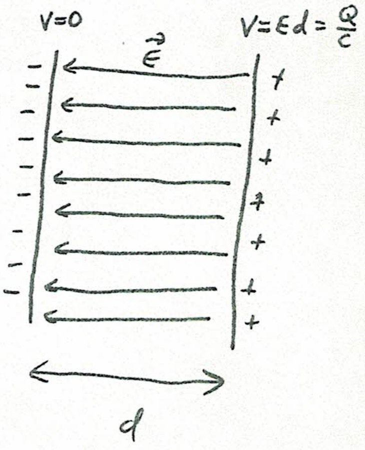
SIMPLEST CAPACITOR IS THE PARALLEL PLATE CAPACITOR.

$$\boxed{C = \varepsilon_0 \frac{A}{d}}.$$ (THIS ASSUMES THAT GAP CONTAINS AIR)

$\varepsilon_0 = 8.85 \times 10^{-12} \frac{C}{NM^2}$

THE FIELD IN A PARALLEL PLATE CAPACITOR IS UNIFORM:

(EQUIPOTENTIAL LINES ARE VERTICAL)



THERE IS SLIGHT CURVING OF LINES NEAR EDGES. (EDGE EFFECTS)

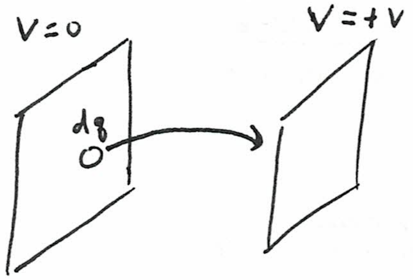
ELECTRIC FIELD: $E = \frac{V}{d}$ (UNIFORM)

(UNIFORM MEANS CONSTANT THROUGHOUT SPACE)

ENERGY

THE ELECTRIC FIELD IN A CAPACITOR STORES POTENTIAL ENERGY.

CAN FIND THIS ENERGY BY LOOKING AT A PARALLEL PLATE CAPACITOR (ALTHOUGH THE RESULT WE'LL GET IS MORE GENERAL).



MOVE A SMALL CHARGE FROM NEGATIVE PLATE TO POSITIVE PLATE. THE CHANGE IN ENERGY OF THE CHARGE IS:

$$dU = V\,dq = \left(\frac{q}{C}\right)dq,$$

WHERE $\pm q$ = TOTAL CHARGE ON EACH PLATE: $q = CV$.

THE TOTAL ENERGY U IS THE ENERGY NEEDED TO MOVE ALL OF THE CHARGE TO THE OTHER PLATE:

$$U = \int dU = \int_0^q \frac{q'}{C}\,dq' = \frac{1}{2C}q^2$$

USING $q = CV$ AGAIN, WE CAN WRITE THIS IN OTHER FORMS:

$$\boxed{U = \frac{q^2}{2C} = \frac{1}{2}CV^2 = \frac{qV}{2}}$$ (ENERGY STORED IN CAPACITOR)

AGAIN: V IS THE VOLTAGE DIFFERENCE BETWEEN THE PLATES

ENERGY DENSITY

ENERGY DENSITY u IS ENERGY PER UNIT VOLUME:

$$u = \frac{\text{ENERGY}}{\text{VOLUME}} \qquad \left(\text{IN } \frac{J}{m^3}\right).$$

AGAIN WE CAN DERIVE THIS FOR A PARALLEL-PLATE CAPACITOR:

PARALLEL PLATE

$$u = \frac{U}{\text{VOL}} = \frac{\frac{1}{2}CV^2}{Ad} = \frac{\frac{1}{2}C(Ed)^2}{Ad} = \frac{1}{2}\left(\frac{\varepsilon_0 \cancel{A}}{\cancel{d}}\right)\frac{E^2\cancel{d^2}}{\cancel{Ad}}$$

$$= \frac{1}{2}\varepsilon_0 E^2$$

SO: $\boxed{u = \frac{1}{2}\varepsilon_0 E^2}$ (ENERGY DENSITY IN ELECTRIC FIELD)

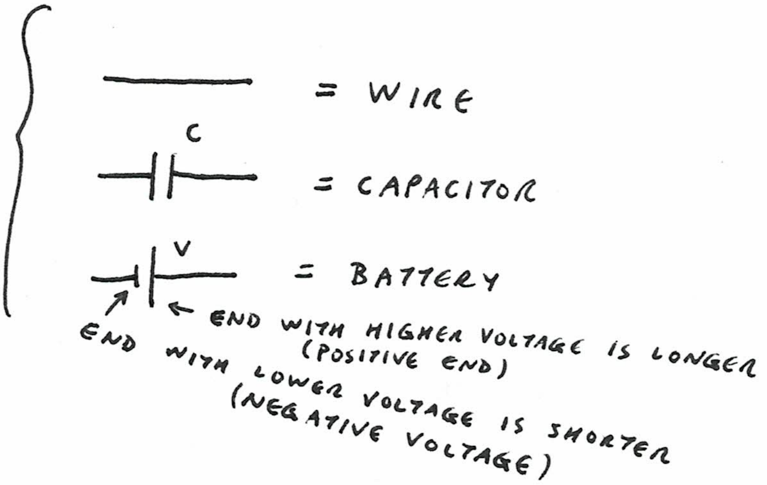
ALTHOUGH WE USED TO PARALLEL PLATE FORMULA TO GET THIS RESULT, IT IS MUCH MORE GENERAL. IT IS TRUE FOR ANY ELECTRIC FIELD (WHETHER INSIDE A CAPACITOR OR NOT)

BE CAREFUL NOT TO CONFUSE:

U = TOTAL ENERGY

u = ENERGY PER VOLUME

DIAGRAMATIC NOTATION:


= WIRE
C
= CAPACITOR
V
= BATTERY
END WITH HIGHER VOLTAGE IS LONGER (POSITIVE END)
END WITH LOWER VOLTAGE IS SHORTER (NEGATIVE VOLTAGE)

DIELECTRICS

IF THERE IS A DIELECTRIC (AN INSULATOR) IN THE GAP OF THE CAPACITOR, JUST REPLACE ε_0 IN ALL FORMULAS BY:

$$\boxed{\underset{\text{PERMITTIVITY OF THE DIELECTRIC}}{\varepsilon} = \kappa \underset{\text{PERMITTIVITY OF FREE SPACE}}{\varepsilon_0}}$$

GREEK LETTER KAPPA

κ IS THE DIELECTRIC CONSTANT OF THE MATERIAL. IT IS ALWAYS GREATER THAN ONE: $\boxed{\kappa \geq 1}$

VALUES OF κ FOR VARIOUS MATERIALS CAN BE FOUND IN TABLE IN THE BOOK.

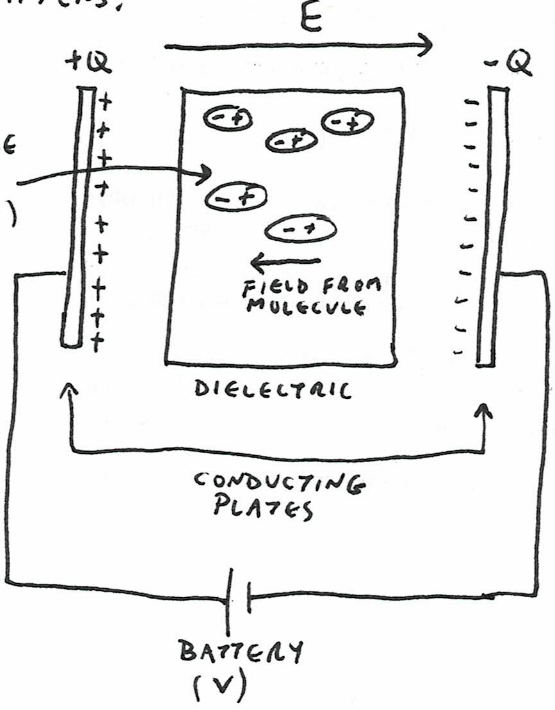
VACUUM: $\kappa = 1.0$ (EXACTLY)

AIR: $\kappa \approx 1.0$ (APPROXIMATELY)

(AT 1 ATM, $\kappa_{AIR} = 1.0006$)

INSERTING DIELECTRIC IN CAPACITOR WILL:
- INCREASE C
- DECREASE E, V
- LEAVE Q UNCHANGED

WHAT HAPPENS:



MOLECULES ARE STRETCHED (OR POLARIZED) BY $\vec{E}_0$. THE SEPARATED CHARGES MAKE THEIR OWN FIELD OPPOSITE TO ORIGINAL $\vec{E}$.

ALTHOUGH Q ON CONDUCTO DOESN'T CHANGE, ITS EFFECT IS WEAKENED OR "SCREENED" BY THE FIELD FROM THE STRETCHED MOLECULES.

THE RESULT IS SMALLER E & V, SO C IS LARGER.

(SINCE $C = \frac{Q}{V}$)

EXAMPLES

1. PARALLEL PLATE CAPACITOR WITH DIELECTRIC $\Rightarrow d = .001\,m,\quad A = 2\times10^{-4}\,m^2,\quad K = 1.3$

a) FIND CAPACITANCE. $C = \frac{\varepsilon A}{d} = \frac{K\varepsilon_0 A}{d} = \frac{(1.3)(8.85\times10^{-12})2}{.001}$

$= \boxed{2.3\times10^{-12}\,F}$

b) IF VOLTAGE IS $V = 30\,V$, FIND Q ON PLATES.

$Q = CV = (2.3\times10^{-12}F)(30\,V) = \boxed{6.9\times10^{-11}\,C}$

c) FIND ELECTRIC FIELD INSIDE CAPACITOR.

E UNIFORM $\Rightarrow E = \frac{V}{d} = \frac{30V}{.001m} = \boxed{30{,}000\ \frac{V}{m}\ (\text{OR } \frac{N}{C})}$

d) FIND V AT A DISTANCE OF $4\times10^{-4}\,m$ FROM NEGATIVE PLATE.

$V = EX = (30{,}000)(4\times10^{-4}) = \boxed{12\,V}$

f) FIND ENERGY DENSITY.

$u = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}K\varepsilon_0 E^2 = \frac{1}{2}(1.3)(8.85\times10^{-12})(3\times10^4)^2$

$= .00518 = \boxed{5.18\times10^{-3}\,\frac{J}{m^3}}$

g) FIND TOTAL ENERGY STORED.

1st METHOD:

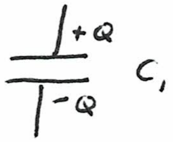
i) ENERGY = (ENERGY DENSITY)(VOLUME) $= u(Ad)$

$= (5.18\times10^{-3})(2\times10^{-4})(10^{-3}) = \boxed{1.04\times10^{-9}\,J}$

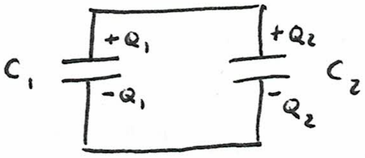
2nd METHOD:

ii) ENERGY $= \frac{1}{2}QV = \frac{1}{2}(6.9\times10^{-11})(30) = \boxed{1.04\times10^{-9}\,J}$

2. $C_1 = 3 \times 10^{-12}$ F, CHARGED TO $Q = 10^{-10}$ C $\Rightarrow$



NOW C_1 IS CONNECTED TO A SECOND CAPACITOR, $C_2 = 2 \times 10^{-12}$ F!



SOME OF THE CHARGE FLOWS FROM C_1 TO C_2.

$\Rightarrow$ FIND Q_1 & Q_2

TWO FACTS:

(i) $Q_1 + Q_2 = Q$ (CHARGE CONSERVATION)

(ii) $V_1 = V_2 \Longrightarrow \dfrac{Q_1}{C_1} = \dfrac{Q_2}{C_2}$

PLUG (i) INTO (ii):

$$\frac{Q_1}{C_1} = \frac{Q - Q_1}{C_2}$$

SOLVE:

$C_2 Q = (Q - Q_1) C_1 \rightarrow (C_2 + C_1) Q_1 = C_1 Q$

$$\Longrightarrow Q_1 = \frac{C_1}{C_1 + C_2} Q = \frac{3 \times 10^{-12}}{5 \times 10^{-12}} (10^{-10}) = \boxed{6 \times 10^{-11} \text{ C}}$$

$$\Rightarrow Q_2 = Q - Q_1 = 10^{-10} - 6 \times 10^{-11} = \boxed{4 \times 10^{-11} \text{ C}}$$

(WE'LL COME BACK TO PROBLEMS LIKE THIS LATER)

DIELECTRIC BREAKDOWN: IF V IS LARGE ENOUGH, THE $\vec{E}$ FIELD CAN STRIP THE DIELECTRIC MOLECULES OF THEIR ELECTRONS, THE ELECTRONS ARE THEN FREE TO CONDUCT, & A SPARK CROSSES THE CAPACITOR.

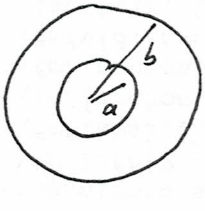
WHEN THIS HAPPENS IN THE ATMOSPHERE (AIR IS A DIELECTRIC), THE RESULT IS LIGHTNING:

(AIR BREAKS DOWN AT $E \sim 10^2 - 10^3 \frac{V}{m}$ DEPENDING ON DENSITY)

NON-PARALLEL PLATE CAPACITORS

SPHERICAL CAPACITOR:

$$\boxed{C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}} = \frac{4\pi\varepsilon a b}{b - a}$$



PROOF:

GAUSS $\Rightarrow$ $$\int \vec{E} \cdot d\vec{A} = 4\pi r^2 E = \frac{q}{\varepsilon}$$

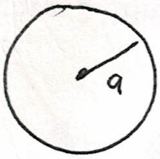
$$E = \frac{q}{4\pi\varepsilon r^2}$$

$$|V| = \left|-\int_a^b E \cdot dr\right| = \left|-\frac{q}{4\pi\varepsilon}\int_a^b \frac{dr}{r^2}\right| = \left|\frac{q}{4\pi\varepsilon}\left(\frac{1}{b} - \frac{1}{a}\right)\right|$$

$$= \frac{q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$C = \frac{q}{V} = \frac{q}{\frac{q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)} \Longrightarrow C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

NOTE: CAN DEFINE CAPACITANCE FOR ISOLATED SPHERE OF RADIUS a, BY TAKING $b \rightarrow \infty$.



$$C = \frac{4\pi a b}{b - a} \approx \frac{4\pi a b}{b} \quad (\text{FOR } b \gg a)$$

$$= 4\pi a$$

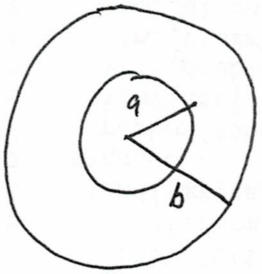
$$\boxed{\text{ISOLATED SPHERE: } C = 4\pi a}$$

CAPACITANCE OF EARTH: $a = 6.37 \times 10^6$ m

$$C = 4\pi a = \boxed{8.0 \times 10^7 \text{ F}}$$

CYLINDRICAL CAPACITOR:

$$\boxed{C = \frac{2\pi \varepsilon L}{\ln\left(\frac{b}{a}\right)}}$$



PROOF: IMAGINE INNER CYLINDER HAS $\sigma = \frac{q}{A}$.

FIELD BETWEEN CYLINDER IS (BY GAUSS):

$$\int \vec{E}\cdot d\vec{A} = 2\pi r L E = \frac{2\pi a L \sigma}{\varepsilon_0}$$

SO: $E = \left(\frac{2\pi a L \sigma}{\varepsilon_0}\right)\cdot\frac{1}{2\pi r L} = \frac{q}{2\pi r \varepsilon_0 L}$

$$|\Delta V| = \left|-\int_a^b \vec{E}\cdot d\vec{r}\right| = \left|-\frac{q}{2\pi\varepsilon_0 L}\int_a^b \frac{dr}{r}\right| = +\frac{q}{2\pi\varepsilon_0 L}\ln\frac{b}{a}$$

$$C = \frac{q}{V} = \frac{q}{\frac{q}{2\pi\varepsilon_0 L}\ln\frac{b}{a}} = \frac{2\pi\varepsilon_0 L}{\ln\frac{b}{a}} \Longrightarrow \text{GIVES RESULT}$$

EXAMPLE: $\kappa = 2$, $L = .02\,m$, $a = .001\,m$, $b = .0015\,m$

VOLTAGE $V = .3V$ IS APPLIED, FIND q & E.

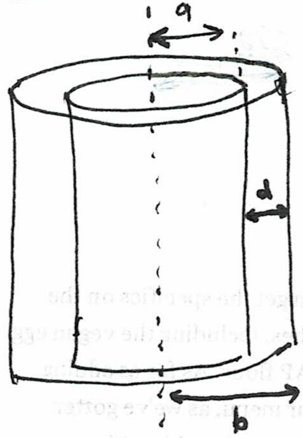
$$C = \frac{2\pi\varepsilon L}{\ln\frac{b}{a}} = \frac{2\pi\kappa\varepsilon_0 L}{\ln\frac{b}{a}} = \frac{2\pi(2)(8.85\times10^{-12})(.02)}{\ln\frac{15}{10}} = \boxed{5.49\times10^{-12}F}$$

$(= 5.49\ pF)$

$$q = CV = (5.49\times10^{-12})(.3) = \boxed{1.65\times10^{-12}C}$$

$$E = \frac{V}{d} = \frac{V}{b-a} = \frac{.3}{.0015-.001} = \frac{.3}{.0005} = \boxed{600\ V/m}$$

EXAMPLE: MYELINATED AXON (HAS OUTER MYELIN SHEATH)



INNER RADIUS: $a = 5 \times 10^{-6}\ m$

THICKNESS OF DIELECTRIC:

$$d = \underbrace{(6 \times 10^{-9}\ m)}_{\text{MEMBRANE}} + \underbrace{(2 \times 10^{-6}\ m)}_{\text{SHEATH (MYELIN)}}$$

$$\simeq 2 \times 10^{-6}\ m$$

OUTER RADIUS:

$$b = a + d = 7 \times 10^{-6}\ m$$

DIELECTRIC CONSTANT: $K = 7.0$

SO: $\varepsilon = K\varepsilon_0 = 7\,(8.85 \times 10^{-12}) = 6.2 \times 10^{-11}\ \frac{C}{Nm^2}$

FOR CYLINDER: $C = \dfrac{2\pi\varepsilon l}{\ln(b/a)}$

SO CAPACITANCE PER UNIT LENGTH:

$$C/l = \frac{2\pi\varepsilon}{\ln(b/a)} = \frac{2\pi\,(6.2 \times 10^{-11})}{\ln\left(\frac{7 \times 10^{-6}}{2 \times 10^{-6}}\right)} = \boxed{3.1 \times 10^{-10}\ F/m}$$

$$= 310\ pF/m$$

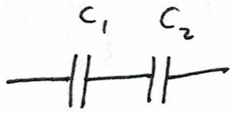
UNMYELINATED AXONS:

$$\frac{C}{l} \sim 3 \times 10^{-7}\ F/m$$

~ 1000 TIMES LARGER

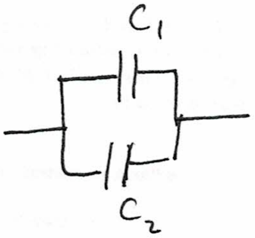
SERIES & PARALLEL CAPACITORS

SERIES:



$$\Rightarrow \left\{ \begin{array}{l} q_1 = q_2 \\ V = V_1 + V_2 \end{array} \right\} \Rightarrow \boxed{\frac{1}{C_{TOTAL}} = \frac{1}{C_1} + \frac{1}{C_2}}$$

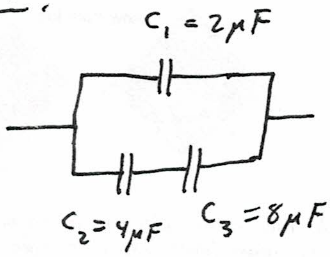
PARALLEL:



$$\Rightarrow \left\{ \begin{array}{l} q = q_1 + q_2 \\ V_1 = V_2 \end{array} \right\} \Rightarrow \boxed{C_{TOTAL} = C_1 + C_2}$$

EXAMPLES:

① a)

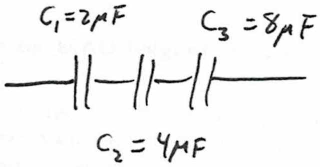


$$C_{TOT} = C_1 + \frac{1}{\frac{1}{C_2} + \frac{1}{C_3}} = 2 + \frac{1}{\frac{1}{4} + \frac{1}{8}}$$

$$= 2 + \frac{1}{3/8} = 2 + \frac{8}{3} = \frac{14}{3}$$

$$= \boxed{4.67\,\mu F}$$

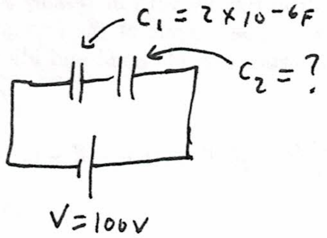
b)



$$C_{TOT} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}} = \frac{1}{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}}$$

$$= \frac{1}{7/8} = \frac{8}{7} = \boxed{1.14\,\mu F}$$

② WHAT CAPACITANCE C_2 IS REQUIRED TO GET 10^{-4} C OF CHARGE ON PLATES?



$$\left. \begin{array}{l} q = C_{TOT} V \\ C_{TOT} = \frac{1}{1/C_1 + 1/C_2} \end{array} \right\} \rightarrow q = \frac{V}{\frac{1}{C_1} + \frac{1}{C_2}}$$

$$\frac{1}{C_1} + \frac{1}{C_2} = \frac{V}{q}$$

$$C_2 = \frac{1}{\frac{V}{q} - C_1} = \frac{1}{\frac{100}{10^{-4}} - \frac{1}{2\times10^{-6}}} = \frac{1}{10^6 - .5\times10^{+6}} = \frac{1}{.5\times10^{+6}}$$

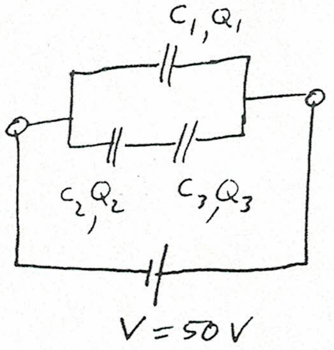
$$= 2\times10^{-6}\,F = \boxed{2\mu F}$$

3. HOW MUCH CHARGE IS ON EACH PLATE OF CAPACITORS FROM EXAMPLE 1, PART a WHEN A VOLTAGE OF 50 V IS APPLIED?

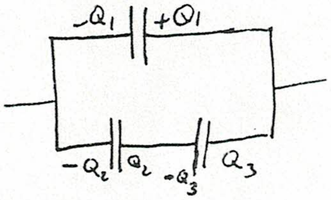
RECALL:

$C_1 = 2\mu F$ $C_2 = 4\mu F$

$C_3 = 8\mu F$



$C_{TOT} = 4,67\mu F$



CHARGE CONSERVATION $\Longrightarrow$ $\boxed{Q_2 = Q_3}$

APPLY $Q = CV$:

i) $\boxed{Q_1 + Q_3 = C_{TOT} V}$

ii) $\boxed{Q_1 = C_1 V}$

so: $Q_1 = C_1 V = (2\times10^{-6})(50) = \boxed{10^{-4} C}$

$Q_3 = C_{TOT}V - Q_1 = (4.67\times10^{-6})(50) - 10^{-4} = .0001335$

$= \boxed{1,335\times10^{-4}\ C}$

$Q_2 = Q_3 = \boxed{1,335\times10^{-4} C}$

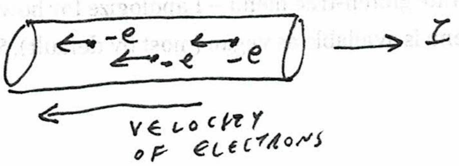
SUMMARY:

SERIES:
1. CHARGES EQUAL
2. VOLTAGES ADD
3. INVERSES OF C ADD

PARALLEL:
1. VOLTAGES EQUAL
2. CHARGES ADD
3. CAPACITANCES ADD

CH. 25 – ELECTRIC CURRENT & RESISTANCE

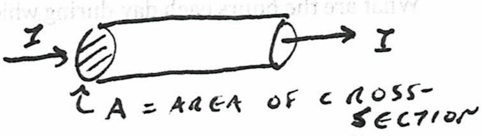
CURRENT: $\underbrace{i = \frac{dq}{dt}}_{\text{CHARGE PER TIME}}$, MEASURED IN AMPS ($1\,A = 1\,\frac{C}{s}$)


-e
-e
-e
-e
I
VELOCITY OF ELECTRONS

IF THE WIRE HAS CROSS-SECTION OF AREA A, THE **CURRENT DENSITY** IS $J = i/A$. (UNITS = A/m^2)

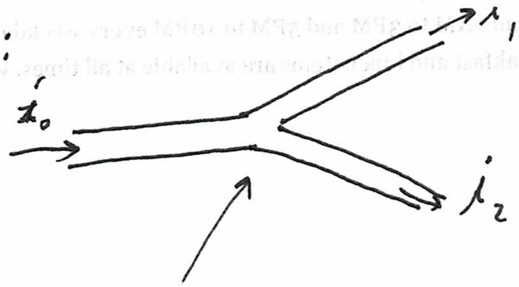
(CURRENT PER ARE)

CONVERSELY, $i = \int \vec{J} \cdot d\vec{A}$.


I
I
A = AREA OF CROSS-SECTION

CURRENT IS CONSERVED:

$$\boxed{i_0 = i_1 + i_2}$$



CURRENT INTO BRANCH POINT (NODE OR VERTEX) MUST EQUAL CURRENT OUT.

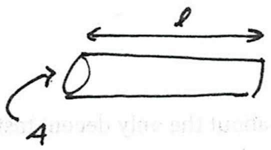
IF A CONDUCTOR HAS A VOLTAGE DIFFERENCE V BETWEEN ITS ENDS, THEN A CURRENT WILL FLOW. THE SIZE OF THE CURRENT WILL BE PROPORTIONAL TO THE VOLTAGE DIFFERENCE. THE PROPORTIONALITY CONSTANT IS CALLED THE RESISTANCE, R:

$$\boxed{\text{OHM'S LAW: } V = iR}$$

UNITS OF R: OHMS, $1\,\Omega = 1\,\frac{V}{A} = 1\,\frac{V\cdot s}{C}$.

RESISTANCE DEPENDS ON MATERIAL, SIZE & SHAPE OF CONDUCTOR, & TEMPERATURE:

$$\boxed{R = \frac{\rho l}{A}}$$



ρ = RESISTIVITY, DEPENDS ON MATERIAL & TEMP.

(TABLE 26-1 p. 689) $\nearrow$ MATERIAL

($\rho - \rho_0 = \alpha \rho_0 (T - T_0)$) $\uparrow$ TEMP.

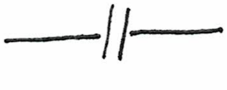
AS CURRENT FLOWS, COLLISIONS BETWEEN ELECTRONS & ATOMS CAUSE CURRENT TO LOSE ENERGY. THE LOST KE GETS TURNED INTO HEAT.

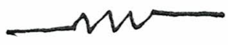
POWER: $$\boxed{P = iV = i^2R = \frac{V^2}{R}}$$

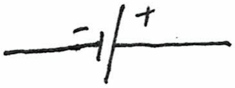
$= \text{RATE OF KE } \underline{\text{LOSS}}$

$= \text{RATE OF HEAT } \underline{\text{GAIN}}$

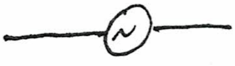
CIRCUIT DIAGRAM SYMBOLS:


= CAPACITOR


= RESISTOR

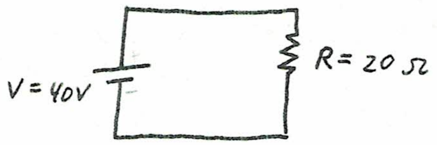


= BATTERY (DC POWER SUPPLY)


= AC POWER SUPPLY

EXAMPLES

1\.



a) WHICH WAY DO CURRENT & ELECTRONS FLOW?

CURRENT → CLOCKWISE
ELECTRONS → COUNTERCLOCKWISE

THINK OF BATTERY LIKE PUMP CARRYING WATER UP MOUNTAIN. THE WATER WANTS TO FLOW BACK TO LOW PE (LOW V).

b) HOW LARGE IS THE CURRENT?

$$i = \frac{V}{R} = \frac{40\ \text{V}}{20\ \Omega} = \boxed{2\ \text{A}}$$

c) HOW MANY ELECTRONS FLOW PER MINUTE?

$$q = i\,\Delta t = \left(2\ \tfrac{\text{C}}{\text{s}}\right)(60\ \text{s}) = 120\ \text{C}$$

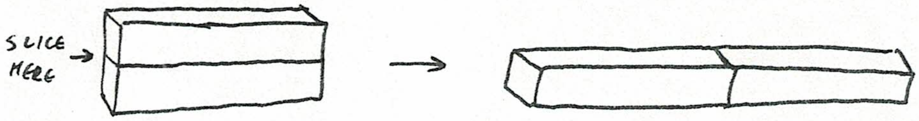
$$\#\ \text{OF ELECTRONS} = \frac{q}{e} = \frac{120}{1.6\times10^{-19}} = \boxed{7.5\times10^{20}}$$

d) HOW MUCH HEAT IS GENERATED IN RESISTOR PER MINUTE?

$$P = i^2 R = (2\ \text{A})^2(20\ \Omega) = 80\ \text{W}$$

$$\text{HEAT} = P\,\Delta t = \left(80\ \tfrac{\text{J}}{\text{s}}\right)(60\ \text{s}) = \boxed{4800\ \text{J}}$$

2. WE HAVE A SLAB OF MATERIAL WITH RESISTANCE $R = 400\ \Omega$. IF WE SLICE IT IN HALF LENGTHWISE & PUT THE 2 PIECES END TO END, WHAT IS THE NEW RESISTANCE, R'?



ORIGINALLY, $R = \frac{\rho l}{A}$, (ρ = RESISTIVITY OF MATERIAL)

NOW, $\rho' = \rho$, $l' = 2l$, $A' = \frac{1}{2}A$. SO:

$$R' = \frac{\rho' l'}{A'} = \frac{\rho(2l)}{\left(\frac{1}{2}A\right)} = 4\left(\frac{\rho l}{A}\right) = \boxed{4R = 1600\ \Omega}$$

3. HEATING ELEMENT IN ELECTRIC STOVE.

$\rho = 8.2 \times 10^{-5}\ \Omega \cdot m$, $l = 1.1\ m$, $A = 3.1 \times 10^{-6}\ m^2$

$V = 120\ V$

a) FIND HEAT PRODUCED IN 5 MINUTES (300 s)

$$R = \frac{\rho l}{A} = 29\ \Omega$$

$$P = \frac{V^2}{R} = \frac{(120)^2}{29} = 496.6\ W$$

$$HEAT = P\Delta t = \boxed{1.49 \times 10^5\ J}$$

b) FIND CURRENT:

$$i = \frac{V}{R} = \frac{120}{29} = \boxed{4.14\ A}$$

4. A FUSE IS RATED AT 20 W. WE FIND THAT IT BURNS OUT AT 100 V.

a) FIND i & R.

$$P = iV \rightarrow i = \frac{P}{V} = \frac{20}{100} = \boxed{.2\ A}$$

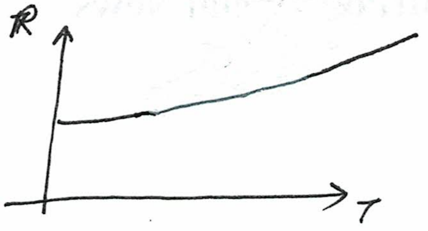
$$R = \frac{V}{i} \rightarrow R = \frac{100}{.2} = \boxed{500\ \Omega}$$

$$\left(\text{OR: } P = i^2 R \rightarrow R = \frac{P}{i^2} = \frac{20}{(.2)^2} = 500\ \Omega\right)$$

b) FUSE IS CONNECTED INSIDE A DEVICE, SO THAT TOTAL R OF DEVICE + FUSE IS CHANGED. NOW V = 75 V. WHAT IS MAX. CURRENT BEFORE BLOW OUT?

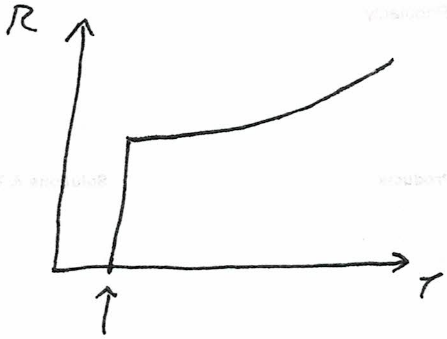
$$i = \frac{P}{V} = \frac{20}{75} = \boxed{.267\ A}$$

FOR MOST RESISTORS, R INCREASES SLOWLY WITH INCREASING TEMP. !



THIS BECAUSE AT HIGH T, THE MOLECULES VIBRATE MORE, MAKING A LARGER TARGET FOR COLLISIONS.

FOR A SUPERCONDUCTOR, THINGS LOOK DIFFERENT!
(KAMMERLINGH ONNES, Hg, 1911)



T_c = CRITICAL TEMP.

CONVENTIONAL SUPERCONDUCTORS:

$$T_c \leq 23 \text{ K}$$

HIGH T_c SUPERCONDUCTORS (1986):

$$T_c = 70 - 100 \text{ K}$$

PROPERTIES:

1. NO POWER IS DISSIPATED BY CURRENTS
2. MAGNETIC FLUX IS EXCLUDED (LEVITATING TRAINS, SQUIDS)
3. IF CURRENT IS TOO LARGE OR MAGNETIC FIELD IS TOO STRONG, SUPERCONDUCTIVITY IS DESTROYED.

HOW FAST DO THE ELECTRONS IN THE CURRENT MOVE?

V_d = DRIFT VELOCITY = AVERAGE ELECTRON SPEED DUE TO VOLTAGE (BEYOND TYPICAL THERMAL MOTION)

TYPICALLY, $V_d \sim 10^{-5} - 10^{-2} \frac{m}{s}$ IN GOOD CONDUCTORS.

HOW IS V_d RELATED TO CURRENT DENSITY?

$$J = \frac{\text{CURRENT}}{\text{AREA}} = \frac{\text{CHARGE}}{\text{TIME} \cdot \text{AREA}} = \frac{q}{tA} = \frac{q}{(XA)} \cdot \frac{X}{t}$$

$$= \frac{q}{\text{VOL.}} \cdot V_d = \frac{Ne}{\text{VOL}} \cdot V_d .$$

SO: $\boxed{J = n e V_d}$, WHERE $n = \left(\begin{array}{l} \# \text{ OF ELECTRONS} \\ \text{PER VOLUME} \end{array} \right)$

= ELECTRON DENSITY

EXAMPLE: 1 V APPLIED ACROSS 5 cm LONG COPPER WIRE ($n = 8.5 \times 10^{28} m^{-3}$) OF RADIUS $r = .1 mm$. WHAT IS DRIFT VELOCITY? ($\rho_{cu} = 1.7 \times 10^{-8} \Omega m$)

- AREA: $A = \pi r^2 = \pi (10^{-4} m)^2 = 3.14 \times 10^{-8} m^2$
- RESISTANCE: $R = \frac{\rho l}{A} = \frac{(1.7 \times 10^{-8} \Omega m)(.05 m)}{3.14 \times 10^{-8} m^2} = 2.7 \times 10^{-2} \Omega$
- CURRENT: $i = V/R = \frac{1 V}{2.7 \times 10^{-2} \Omega} = 36.9 A$
- CURRENT DENSITY: $J = i/A = \frac{36.9}{3.16 \times 10^{-6}} = 1.2 \times 10^{9} \frac{A}{m^2}$
- DRIFT SPEED:

$$V_d = \frac{J}{ne} = \frac{1.2 \times 10^{9}}{(8.5 \times 10^{28})(1.6 \times 10^{-19})} = \boxed{.088 \frac{m}{s}}$$

ALTERNATE VERSION OF OHM'S LAW

$$V = IR \longrightarrow \underbrace{\frac{V}{l}}_{E} = \frac{IR}{l} \longrightarrow E = \underbrace{\left(\frac{I}{A}\right)}_{J}\underbrace{\left(\frac{RA}{l}\right)}_{\rho}$$

SO $E = J\rho$, OR EQUIVALENTLY:

$$\boxed{\vec{J} = \sigma \vec{E}}$$, WHERE $\sigma = \frac{1}{\rho}$ = CONDUCTIVITY

(IN $\Omega^{-1} m^{-1} \equiv$ MHOS)

↑ (NOTE THAT BOTH SIDES ARE NOW VECTOR)

EXAMPLE: FIND ELECTRIC FIELD IN LAST EXAMPLE.

CAN DO TWO WAYS:

(i) $\sigma = \frac{1}{\rho} = \frac{1}{1.7 \times 10^{-8}} = 5.9 \times 10^{7}\ \Omega^{-1}$

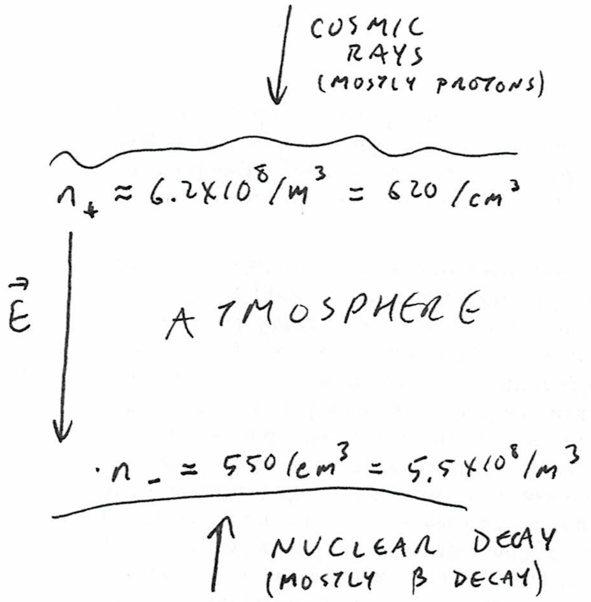
$$E = \frac{J}{\sigma} = \frac{1.2 \times 10^{9}\ \frac{A}{m^2}}{5.9 \times 10^{7}\ \Omega^{-1} m^{-1}} = \boxed{20.3\ \frac{V}{m}}$$

DIFFERENCE DUE TO ROUNDING

(ii) $E = \frac{V}{l} = \frac{1\,V}{.05\,m} = \boxed{20.0\ \frac{V}{m}}$

↑ LENGTH OF WIRE

EXAMPLE


COSMIC RAYS (MOSTLY PROTONS)
$n_+ \approx 6.2\times10^8/m^3 = 620/cm^3$
$\vec{E}$
ATMOSPHERE
$n_- = 550/cm^3 = 5.5\times10^8/m^3$
NUCLEAR DECAY (MOSTLY β DECAY)

POSITIVE CHARGES ACCUMULATE AT TOP OF ATMOSPHERE, NEGATIVE AT BOTTOM, RESULT IS $\vec{E} \approx 100 \frac{V}{m}$ POINTING DOWNWARD.

RESISTIVITY OF AIR:

$$\rho \approx 3.7\times10^{13}\,\Omega m$$

a) FIND CURRENT DENSITY.

$$J = \sigma E \leftarrow \left(\sigma = \frac{1}{\rho} = 2.7\times10^{-14}\,\Omega^{-1}m^{-1}\right)$$

$$= (2.7\times10^{-14})\,100$$

$$= \boxed{2.7\times10^{-12}\ A/m^2}$$

b) FIND DRIFT SPEEDS OF + & − IONS, (ASSUME HALF OF J IS DUE TO EACH TYPE OF ION)

$$v_{d\pm} = \frac{J/2}{n_\pm e} = \begin{cases} \dfrac{2.7\times10^{-12}/2}{(6.2\times10^8)(1.6\times10^{-19})} = \boxed{.0136\ \frac{m}{s}} \\ \dfrac{2.7\times10^{-12}/2}{(5.5\times10^8)(1.6\times10^{-19})} = \boxed{.0153\ \frac{m}{s}} \end{cases}$$

c) HOW MUCH CHARGE COLLECTS IN 100 km² OF UPPER ATMOSPHERE PER DAY? (DUE TO CURRENT FLOW OF PART b)

$$t = (24\ HR.)\left(3600\ \frac{s}{HR}\right) = 86{,}400\,s \quad , \quad A = 10^8\ m^2$$

$$|q| = i\cdot t = JAt = \left(2.7\times10^{-12}\ \frac{A}{m^2}\right)(10^8\ m^2)(86{,}400\,s)$$

$$= 23.3\ C \rightarrow \boxed{\begin{array}{l} q = -23.3\ C \text{ IN ATMOSPHERE} \\ q = +23.3\ C \text{ AT GROUND} \end{array}}$$

THIS PARTIALLY CANCELS n_+ ABOVE. TO MAINTAIN ELECTRICAL NEUTRALITY OVER TIME → LIGHTNING.

CH. 26 - DC ELECTRIC CIRCUITS

BATTERY: CONVERTS POTENTIAL ENERGY IN CHEMICAL BONDS INTO $\underbrace{\text{VOLTAGE}}_{\text{ELECTROSTATIC PE}}$ & $\underbrace{\text{CURRENT}}_{\text{KE}}$.

THE VOLTAGE PRODUCED BY THE BATTERY IS CALLED THE ELECTROMOTIVE FORCE (EMF), $\mathcal{E}$.

NOTE: $\mathcal{E}$ = VOLTAGE, NOT ELECTRIC FIELD.

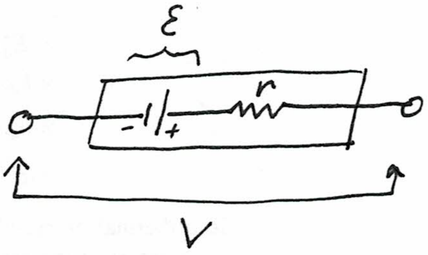
CAN VIEW $\mathcal{E}$ AS WORK DONE BY BATTERY PER UNIT CHARGE:

$$\boxed{\mathcal{E} = \frac{W}{q}}$$

NOTE THAT REAL BATTERIES HAVE INTERNAL RESISTANCE, r, SO TERMINAL VOLTAGE V IS ALWAYS LESS THAN $\mathcal{E}$:

$$\boxed{V = \mathcal{E} - Ir}$$

REAL BATTERY (V) IDEAL BATTERY ($\mathcal{E}$)



SIGNS OF VOLTAGES

(i) BATTERIES: $\begin{cases} V \text{ IS POSITIVE GOING FROM } - \text{ TO } + \text{ TERMINAL} \\ V \text{ IS NEGATIVE GOING FROM } + \text{ TO } - \text{ TERMINAL} \end{cases}$

CURRENT LOSING ENERGY

(ii) RESISTORS: $\begin{cases} V \text{ IS NEGATIVE MOVING IN DIRECTION OF } i. \\ V \text{ IS POSITIVE MOVING AGAINST } i. \end{cases}$

CURRENT GAINING ENERGY

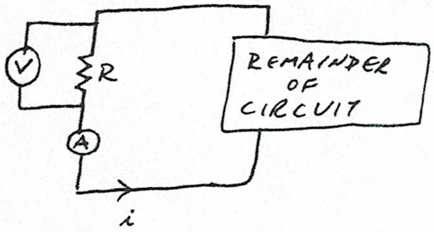
METERS

I. AN **AMMETER** OR **GALVANOMETER** MEASURES CURRENT. IT SHOULD HAVE LOW RESISTANCE & BE IN SERIES WITH THE OBJECT WHOSE CURRENT IS TO BE MEASURED.

CIRCUIT SYMBOL: —Ⓐ— OR —Ⓖ—

II. A **VOLTMETER** MEASURES VOLTAGE **DIFFERENCES** BETWEEN POINTS. IT SHOULD HAVE HIGH RESISTANCE & BE CONNECTED IN PARALLEL TO THE OBJECT BEING MEASURED. SYMBOL: —Ⓥ—

EXAMPLE:

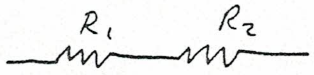


III. **OHMMETER** MEASURES RESISTANCE. CONNECTED IN PARALLEL. CREATES KNOWN VOLTAGE DIFFERENCE, THEN MEASURES CURRENT DRAWN.

NOTE: IN CIRCUIT ABOVE, OHMMETER WOULD NOT MEASURE R ALONE, BUT R IN PARALLEL WITH REST OF CIRCUIT.

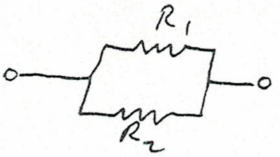
EQUIVALENT RESISTANCES:

1. SERIES: $R_{EQ} = R_1 + R_2$



$\left(\begin{array}{l} I_1 = I_2 \\ V = V_1 + V_2 \end{array} \right)$

2. PARALLEL: $\frac{1}{R_{EQ}} = \frac{1}{R_1} + \frac{1}{R_2}$



$\left(\begin{array}{l} I = I_1 + I_2 \\ V_1 = V_2 \end{array} \right)$

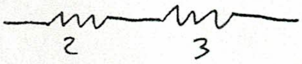
KIRCHOFF'S RULES

1. JUNCTION RULE: $\sum_{IN} I = \sum_{OUT} I$ (AT EACH JUNCTION)
   - THIS IS JUST CURRENT CONSERVATION.
2. LOOP RULE: $\sum V = 0$ AROUND ANY CLOSED LOOP
   - THIS IS JUST ENERGY CONSERVATION
   - BE VERY CAREFUL OF SIGNS.

EXAMPLES

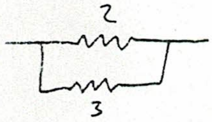
1. FIND TOTAL RESISTANCE OF EACH:

a)



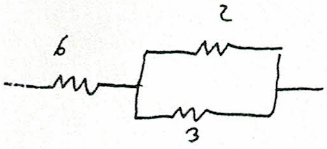
$= \boxed{5}\,\Omega$

b)



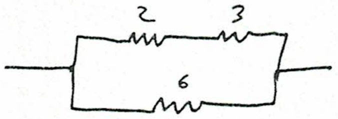
$\Rightarrow \frac{1}{R_{EQ}} = \frac{1}{R_1} + \frac{1}{R_2} \Longrightarrow R_{EQ} = \frac{R_1 R_2}{R_1 + R_2} = \frac{2 \cdot 3}{2+3} = \boxed{1.2}\,\Omega$

c)



$\Rightarrow R_{EQ} = 6 + \frac{1}{\frac{1}{2}+\frac{1}{3}} = 6. + 1.2 = \boxed{7.2\,\Omega}$

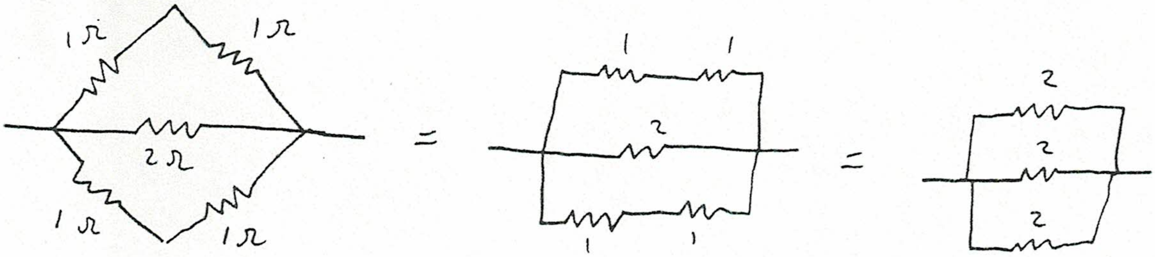
d)



$\Rightarrow R_{EQ} = \frac{1}{\frac{1}{6} + \frac{1}{2+3}} = \frac{1}{\frac{1}{6}+\frac{1}{5}} = \frac{1}{11/30} = \boxed{\frac{30}{11}\,\Omega}$

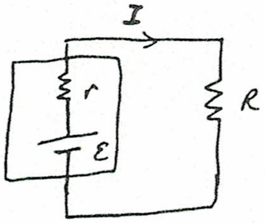
NOTE: SERIES → R INCREASES
PARALLEL → R DECREASES

2.



$$\frac{1}{R_{EQ}} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} \longrightarrow R_{EQ} = \frac{2}{3} = \boxed{.67\ \Omega}$$

3. A BATTERY WITH $\mathcal{E} = 9V$ IS CONNECTED TO A 10 Ω RESISTOR. IF AN .85 A CURRENT IS PRODUCED, FIND THE BATTER TERMINAL VOLTAGE (V) & INTERNAL RESISTANCE (r).



$$R_{EQ} = r + 10$$

$$I = \frac{\mathcal{E}}{R_{EQ}} = \frac{\mathcal{E}}{r+R}$$

SOLVE FOR r: $r = \frac{\mathcal{E}}{I} - R = \frac{9}{.85} - 10 = \boxed{.588\ \Omega}$

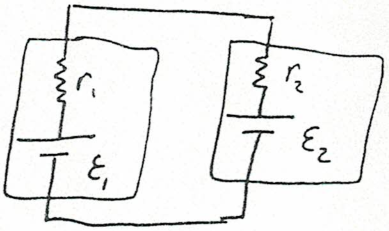
$$V = \mathcal{E} - Ir = 9 - (.85)(.588) = \boxed{8.50\ V}$$

NOTE: 1. WILL ALWAYS HAVE $V \leq \mathcal{E}$.

2. COMPARE: $\mathcal{E} = I(r+R)$

$$V = IR$$

4. CONNECT 2 BATTERIES:



LET $r_1 = .05\,\Omega$, $r_2 = .1\,\Omega$

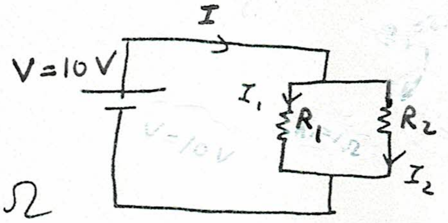
$\mathcal{E}_1 = 2\,V$, $\mathcal{E}_2 = 1.5\,V$.

FIND I.

$$0 = \sum V = \mathcal{E}_1 - (r_1 + r_2) - \mathcal{E}_2$$

$$I = \frac{\mathcal{E}_1 - \mathcal{E}_2}{r_1 + r_2} = \frac{2 - 1.5}{.05 + .1} = \frac{.5}{.15} = \boxed{3.33\ A}$$ (CLOCKWISE)

5. FIND THE CURRENT IN EACH PART OF THE CIRCUIT ($R_1 = 1\,\Omega$, $R_2 = 2\,\Omega$)



(ASSUME $r = 0$)

i) TO FIND I:

$$R_{TOTAL} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{1}{1 + \frac{1}{2}} = \frac{1}{(3/2)} = \frac{2}{3}\,\Omega$$

$$I = \frac{V}{R_{TOTAL}} = \frac{10}{(2/3)} = \frac{30}{2} = \boxed{15\ A}$$

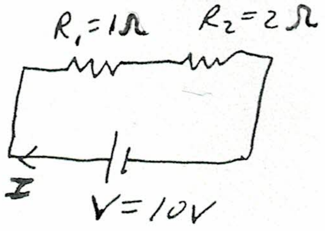
ii) TO FIND I_1: $I_1 = \frac{V}{R_1} = \frac{10\,V}{1\,\Omega} = \boxed{10\ A}$

iii) TO FIND I_2: $I_2 = \frac{V}{R_2} = \frac{10\,V}{2\,V} = \boxed{5\ A}$

-OR-

$$I_2 = I - I_1 = \boxed{5A}$$

6. FIND V & I FOR EACH RESISTOR:



i) $I_1 = I_2 = I = \frac{V}{R_1 + R_2} = \frac{10}{1+2} = \boxed{3.33\ A}$

ii) $V_1 = I_1 R_1 = (3.33)(1) = \boxed{3.33\ V}$

iii) $V_2 = I_2 R_2 = (3.33)(2) = \boxed{6.67\,V}$

-OR-

$$V_2 = V - V_1 = \boxed{6.67\ V}$$

MORE EXAMPLES: CHAPTER 26

⑦

EXAMPLE 7 Currents and Voltages in a Multiloop Circuit

(a) Find the currents in Fig. 20-22a. (b) Find the voltage across the terminals of the 15 V battery, V_{ba}. (The 1.0 Ω resistor is the battery's internal resistance.)

SOLUTION (a) First we label the currents in such a way that Kirchhoff's first rule is satisfied at each branch point. This is indicated in Fig. 20-22b. Next we apply Kirchhoff's second rule to the two loops, starting in the upper left-hand corner of each loop and proceeding clockwise. For loop 1, we have

$$\Sigma V = 0$$

$$-15.0\text{ V} + I_1(1.0\ \Omega) + I_1(8.0\ \Omega) - I_2(20.0\ \Omega) - I_2(4.0\ \Omega) + 5.0\text{ V} = 0$$

We may simplify this equation and drop the units, remembering that the current is in amps:

$$9.0I_1 - 24I_2 = 10.0 \qquad (20\text{-}12)$$

For loop 2, we get

$$\Sigma V = 0$$

$$-5.0 + 24I_2 + 18(I_1 + I_2) = 0$$

or

$$18I_1 + 42I_2 = 5.0 \qquad (20\text{-}13)$$

Finally, we solve Eqs. 20-12 and 20-13 for I_1 and I_2. Multiplying Eq. 20-12 by 2 and subtracting from Eq. 20-13, we obtain

$$90I_2 = -15.0$$

$$I_2 = -0.17\text{ A}$$

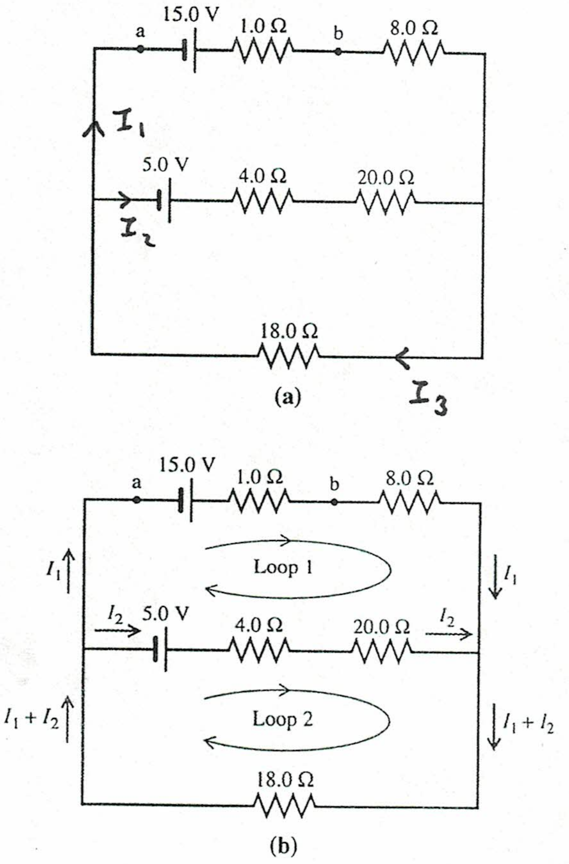
Returning to Eq. 20-12, we find

$$9.0I_1 - 24(-0.17) = 10.0$$

$$9.0I_1 = 6.0$$

$$I_1 = 0.67\text{ A}$$

$$I_3 = I_1 + I_2 = .5\text{ A}$$



(a)

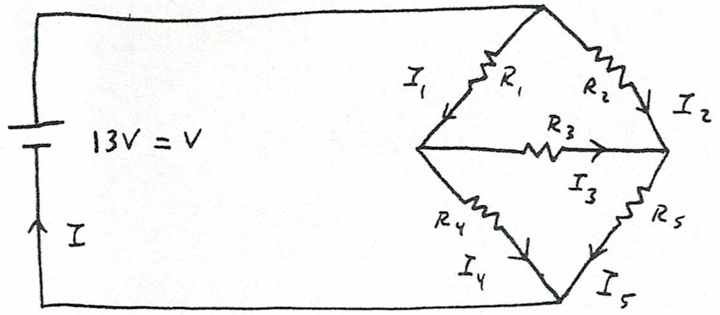
(b)

Fig. 20-22

(b) Now we can calculate the voltage drop across the 15 V battery:

$$V_{ba} = -(0.67\text{ A})(1.0\ \Omega) + 15.0\text{ V}$$
$$= 14.3\text{ V}$$

⑧



$R_1 = R_2 = R_3 = R_4 = 1\,\Omega$

$R_5 = 2\,\Omega$

$V = 13\,V$

JUNCTION RULE GIVES US:

$$I = I_1 + I_2$$
$$I_4 = I_1 - I_3$$
$$I_5 = I_2 + I_3$$

LOOP RULE GIVES US;

$$V - I_1 R_1 - I_4 R_4 = 0$$
$$-I_2 R_2 - I_5 R_5 + V = 0$$
$$-I_1 R_1 - I_3 R_3 + I_2 R_2 = 0$$

PLUGGING THE FIRST THREE EQUATIONS INTO THE 2nd THREE;

$$V - I_1 R_1 - (I_1 - I_3) R_4 = 0$$
$$-I_2 R_2 - (I_2 + I_3)(R_5) + V = 0$$
$$-I_1 R_1 - I_3 R_3 + I_2 R_2 = 0$$

SOLVE THIRD EQUATION & PLUG RESULT INTO FIRST TWO, WE'RE LEFT WITH:

$$V = (2\Omega) I_1 - (1\Omega) I_3$$
$$V = (3\Omega) I_1 + (5\Omega) I_3$$

(AFTER PLUGGING IN VALUES OF RESISTORS)

COMBINE THESE TWO EQUATIONS TO ELIMINATE I_3. RESULT:

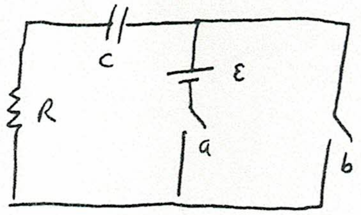
$$78\,V = I_1 (13\,\Omega) \implies \boxed{I_1 = 6A}$$

WORKING BACKWARD THROUGH PREVIOUS EQUATIONS WE OBTAIN REST OF CURRENTS:

$$\boxed{\begin{aligned} I_3 &= -1A \\ I_2 &= 5A \\ I_4 &= 7A \\ I_5 &= 4A \end{aligned}}$$

$$\boxed{I = 11A}$$

RC CIRCUITS



i) WHEN SWITCH a IS CLOSED:

CURRENT FLOWS THROUGH R. IT CAN'T CROSS CAPACITOR, SO CHARGE BUILDS UP ON PLATES

ii) WHEN b IS CLOSED & a OPENED:

THE CHARGE DISPERSES, CREATING ANOTHER CURRENT THROUGH RESISTOR. CHARGE ON PLATES DECAYS OVER TIME.

DEFINE TIME CONSTANT: $\tau \xleftarrow{\text{GREEK TAU}} = RC$ (UNITS = SECONDS)

i) CHARGING: $\overset{\swarrow \text{VOLTAGE ON CAPACITOR}}{V} = \mathcal{E}(1 - e^{-t/\tau})$

$$q = CV \Longrightarrow q = C\mathcal{E}(1 - e^{-t/\tau})$$

$$I = \frac{dq}{dt} \Longrightarrow I = \frac{C\mathcal{E}}{\tau} e^{-t/\tau} = \frac{\mathcal{E}}{R} e^{-t/\tau}$$

ii) DISCHARGING: $V = \mathcal{E} \cdot e^{-t/\tau}$

$$q = C\mathcal{E} e^{-t/\tau}$$

$$I = -\frac{C\mathcal{E}}{\tau} e^{-t/\tau} = \underset{\uparrow}{-}\frac{\mathcal{E}}{R} e^{-t/\tau}$$

(i FLOWS IN OPPOSITE DIRECTION NOW)

EXAMPLE: A CAPACITOR IS CHARGED BY A 9V BATTERY. ASSUME $r=0$

BATTERY, $R = 100\,\Omega$, $C = 3\times10^{-6}\,F$

a) WHAT IS CHARGE ON CAPACITOR AFTER 2×10^{-4} s?

$$q = C\mathcal{E}(1-e^{-t/\tau}) = (2.7\times10^{-5})(1-e^{-.667}) = \boxed{1.31\times10^{-5}\,C}$$

b) WHAT IS CHARGE AFTER A LONG TIME ($t\to\infty$)?

$$q = C\mathcal{E} = \boxed{2.7\times10^{-5}\,C}$$

c) HOW LONG DOES IT TAKE THE CAPACITOR TO DISCHARGE ½ ITS CHARGE?

$$q = C\mathcal{E}\,e^{-t/\tau} = q_0\,e^{-t/RC}$$

$$q = \tfrac{1}{2}q_0 \longrightarrow \tfrac{1}{2} = e^{-t/RC} \longrightarrow \ln\tfrac{1}{2} = -\frac{t}{RC}$$

SO, $t = -RC\ln\tfrac{1}{2} = +RC\ln 2 = \boxed{2.08\times10^{-4}\,s}$

d) WHAT IS THE VOLTAGE ACROSS THE CAPACITOR 10^{-4} s AFTER DISCHARGE BEGINS?

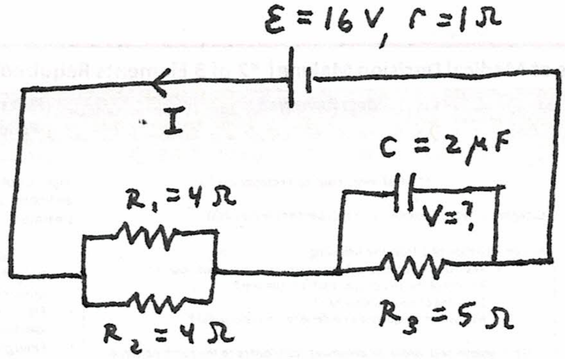
$$V(t) = \frac{q(t)}{C} = \mathcal{E}\,e^{-t/\tau} = (9)\,e^{-\frac{10^{-4}}{3\times10^{-4}}} = \boxed{6.45\ V}$$

e) CURRENT AFTER 10^{-4} s?

$$I(t) = \frac{V(t)}{R} = \frac{6.45\,V}{100\,\Omega}$$

$$= \boxed{-6.45\times10^{-2}\,A}$$

②



FIND THE VOLTAGE V ACROSS THE CAPACITOR AND THE CHARGE ON ITS PLATES,
$\underbrace{\text{CHARGE}}_{Q}$

FIRST FIND TOTAL RESISTANCE OF CIRCUIT:

$$R = r + R_3 + \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}} = 1 + 5 + \frac{1}{\frac{1}{4}+\frac{1}{4}}$$

$$= 1 + 5 + 2 = 8\,\Omega .$$

THE CAPACITANCE DOESN'T AFFECT THE STEADY-STATE DC CURRENT, SO:

$$I = \frac{\mathcal{E}}{R} = \frac{16\,V}{8\,\Omega} = 2\,A$$

SO THE VOLTAGE ACROSS THE 5 Ω RESISTOR IS:

$$IR_3 = (2\,A)(5\,\Omega) = 10\,V .$$

BUT C IS IN PARALLEL WITH R_3, SO THEY MUST HAVE THE SAME VOLTAGE:

$$\boxed{V = 10\,V}$$

CHARGE: $Q = CV = (2\times10^{-6}\,F)(10\,V) = \boxed{2\times10^{-5}\,F}$

$(= 20\,\mu F)$

NOTE THAT VOLTAGE ACROSS C DOES NOT EQUAL THE BATTERY VOLTAGE. THEY ARE NOT IN PARALLEL

CH. 27 - MAGNETIC FIELDS

STATIC CHARGES → ELECTRIC FIELD: $\vec{E}$
(COULOMB'S LAW)

MOVING CHARGES → MAGNETIC FIELD: $\vec{B}$
(CURRENTS)
(BIOT-SAVART LAW)

INVERSE SQUARE LAWS

UNITS OF $\vec{B}$: TESLAS (T) OR GAUSS (G)

$$\left(1\ T = 1\ \frac{N}{A\cdot m}\ ,\quad 1\ G = 10^{-4}\ T\right)$$

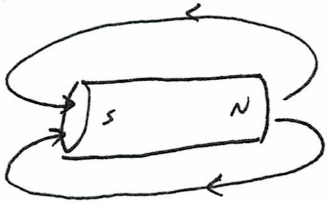
SIZES OF SOME MAGNETIC FIELDS:

EARTH'S MAGNETIC FIELD: $B = 10^{-4}\ T$
BAR MAGNET: $B = 10^{-3} - 10^{-2}\ T$
NEUTRON STAR: $B = 10^{8}\ T$

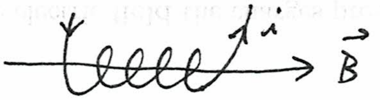
SOURCES OF MAGNETIC FIELDS:

1. **BAR MAGNET:** MADE OF FERROMAGNETIC MATERIALS (IRON, NICKEL, COBALT)

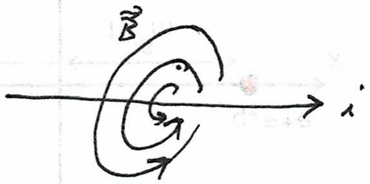
- THE MOVING CHARGES ARE SPINNING NUCLEI
- IN NONMAGNETIC MATERIALS: SPIN DIRECTIONS ARE ORIENTED RANDOMLY, SO FIELDS CANCEL.
- IN FERROMAGNETS: THE ATOMS FORM REGIONS CALLED DOMAINS IN WHICH ALL SPINS ARE ALIGNED IN SAME DIRECTION.
- ABOVE CURIE TEMPERATURE, DOMAINS ARE DESTROYED



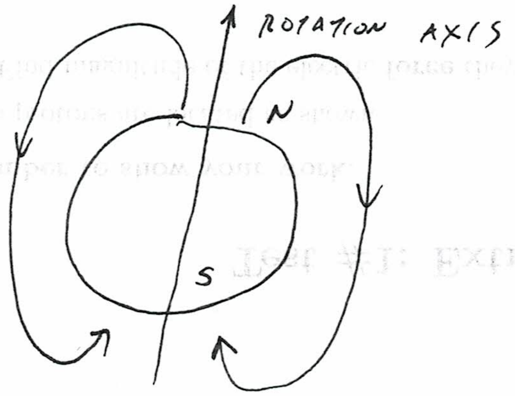
2. **ELECTROMAGNET:** $\vec{B}$ IS PRODUCED BY CURRENT THROUGH WIRE, USUALLY IN SHAPE OF SOLENOID.



A STRAIGHT CURRENT-CARRYING WIRE WILL PRODUCE CIRCULAR FIELDS:

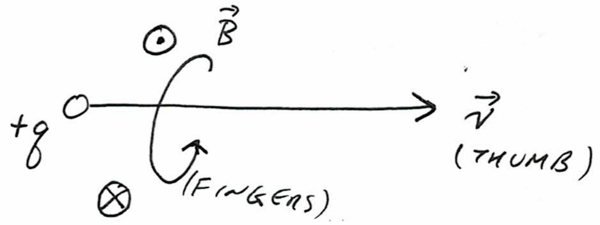


3. **SPINNING PLANET OR STAR:**

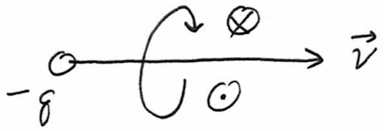


DIRECTION OF $\vec{B}$, GIVEN BY RIGHT-HAND RULE.

FOR POSITIVE CHARGE:

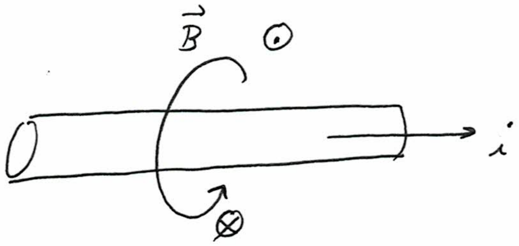

⊙
$\vec{B}$
$+q$
$\vec{v}$
(THUMB)
⊗
(FINGERS)

(FLIP DIRECTION FOR NEGATIVE CHARGE:

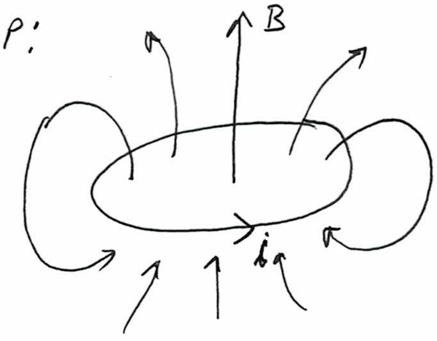


)

FOR CURRENT IN WIRE:



CURRENT LOOP:



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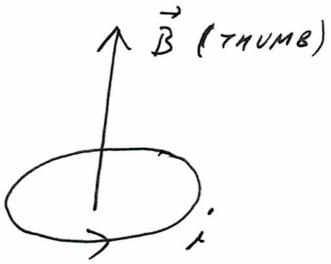
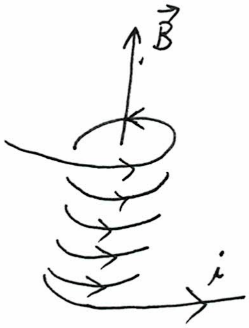

$\vec{B}$ (THUMB)
i

SOLENOID:



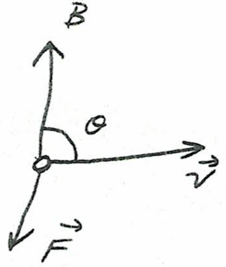
(B ≈ 0 OUTSIDE)

MAGNETIC FORCES

ON CHARGE: $F = qvB\sin\theta \; (= qvB_\perp)$

(DIRECTION $\perp$ TO $\vec{v}$ & $\vec{B}$, GIVEN BY RIGHT-HAND RULE)

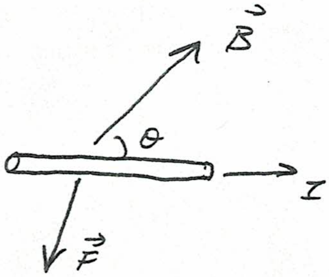
$$\boxed{\vec{F} = q\vec{v}\times\vec{B}}$$



ON CURRENT

$$F = I\ell B\sin\theta \; (= I\ell B_\perp)$$

$$\boxed{\vec{F} = i\,\vec{\ell}\times\vec{B}}$$



EXAMPLES

1. PROTON IS TRAVELLING THROUGH UNIFORM MAGNETIC FIELD, $B = 1.2\times10^{-3}$ T. VELOCITY: $v = 3\times10^{7}\ \frac{m}{s}$.

a) IF v & B ARE $\perp$, FIND FORCE & DESCRIBE PATH.

$$F = qvB\sin\theta = (1.6\times10^{-19})(3\times10^{7})(1.2\times10^{-3})\sin 90$$
$$= \boxed{5.76\times10^{-15}\ N}$$

(PATH IS CIRCULAR)

b) WHAT IS THE RADIUS OF THIS CIRCLE? ($M_{PROTON} = 1.67\times10^{-27}$ kg)

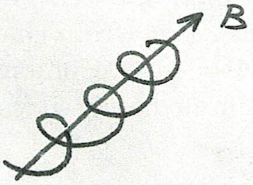
$$F = ma \implies F = m\frac{v^2}{r}$$

$$r = \frac{mv^2}{F} = \frac{(1.67\times10^{-27})(3\times10^{7})^2}{(5.76\times10^{-15})} = \boxed{260.9\ m}$$

2. REPEAT PART a OF LAST EXAMPLE, WITH $\theta = 30°$.

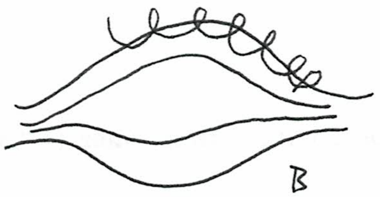
$$F = qvB\sin\theta = (1.6\times10^{-19})(3\times10^{7})(1.2\times10^{-3})\underbrace{\sin 30}_{\frac{1}{2}}$$
$$= \boxed{2.88\times10^{-15}\ N}$$

PATH IS A SPIRAL:



EXAMPLE OF USE OF MAGNETIC FIELD:

CONFINEMENT OF ANTIMATTER IN MAGNETIC BOTTLE.



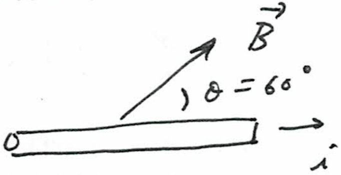
PARTICLES SPIRAL AROUND $\vec{B}$ FIELD LINES.

WANT TO STAY IN REGION OF LOW FIELD.

THEY GET REFLECTED FROM REGIONS OF LARGE $\vec{B}$, I.E. ENDS ACT AS MAGNETIC MIRRORS

MORE EXAMPLES

1. A STRAIGHT WIRE OF LENGTH $l = .2\,m$ CARRIES A CURRENT OF $i = 3\,A$ & IS AT 60° FROM MAGNETIC FIELD, $\vec{B}$ $(B = .05\,T)$.



a) HOW LARGE IS THE FORCE ON WIRE?

$$\vec{F} = i\,\vec{l} \times \vec{B} \Rightarrow F = i l B \sin\theta = (3)(.2)(.05)\sin 60°$$

$$= \boxed{.026\ N}$$

b) DIRECTION OF $\vec{F}$? (OUT OF PAGE)

2. THE WIRE FROM THE LAST EXAMPLE IS BENT INTO A SQUARE LOOP. WHAT IS FORCE NOW?

$$\boxed{F = 0}$$

BUT, THERE IS NOW A TORQUE. ⟶

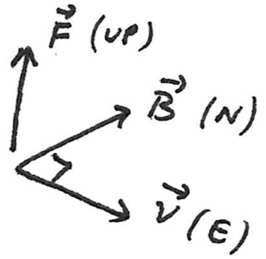
EXAMPLE: NEAR EQUATOR, $\vec{B}_{EARTH}$ IS HORIZONTAL & DUE NORTH, WITH $B = 2.5 \times 10^{-5}\,T = .25\,G$.

a) FIND MAGNITUDE AND DIRECTION OF FORCE ON A PROTON WITH $v = 10^{4}\,\frac{m}{s}$, DUE EAST & HORIZONTAL.

$$F = qvB\sin\theta = (1.6\times10^{-19}\,c)(10^{4}\,\tfrac{m}{s})(2.5\times10^{5}\,T)\sin 90^\circ$$

$$= \boxed{4\times10^{-20}\,N}$$

DIRECTION: VERTICALLY UPWARD.



b) FORCE ON ELECTRON?

$$\boxed{F = 4\times10^{-20}\,N,\ \text{DOWNWARD}}$$

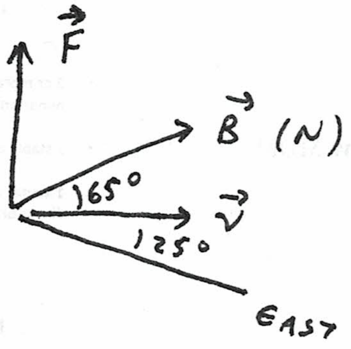
c) PROTON NOW MOVES AT 25° N OF E. FORCE?

$$F = qvB\sin\theta$$

$$= (1.6\times10^{-19})(10^{4})(2.5\times10^{-5})\sin 65^\circ$$

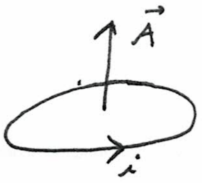
$$= \boxed{3.63\times10^{-20}\,N}$$

(UPWARD)



MAGNETIC DIPOLES

CURRENT LOOP:



$\vec{A}$ = AREA VECTOR
($\perp$ TO LOOP, GIVEN BY RIGHT-HAND RULE)

LOOK HAS A MAGNETIC DIPOLE MOMENT:

$$\boxed{\vec{\mu} = N i \vec{A}} \qquad (\text{IN } A \cdot m^2)$$

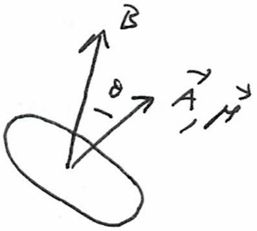
↖ $N = \#$ OF LOOPS

PLACE LOOPS IN $\vec{B}$ FIELD. THEN,

(i) $\vec{B}$ EXERTS TORQUE: $\vec{\tau} = \vec{\mu} \times \vec{B}$ $(\tau = \mu B \sin\theta)$

(ii) LOOPS HAVE POTENTIAL ENERGY WHICH DEPENDS ON ORIENTATION OF LOOP RELATIVE TO $\vec{B}$:

$$\boxed{U(\theta) = -\vec{\mu} \cdot \vec{B}} = -\mu B \cos\theta$$



LOOP WANTS TO ROTATE SO THAT $\vec{\mu}$ IS PARALLEL TO $\vec{B}$.

EXAMPLE: CONSIDER CIRCULAR COIL OF WIRE CARRYING $i = 3A$, IN FIELD $B = 5\ T$.

OF LOOPS: $N = 10$, RADIUS: $r = .1\ m$.

a) MAGNETIC MOMENT OF COIL?

$$\mu = N i A = N i \pi r^2 = (10)(3)\pi(.1)^2$$

$$= \boxed{.942\ Am^2}$$

b) SUPPOSE COIL'S NORMAL VECTOR IS AT 30° TO $\vec{B}$. WHAT IS TORQUE?

$$\vec{\tau} = \vec{\mu} \times \vec{B} \rightarrow \tau = \mu B \sin\theta = (.942)(5)\sin 30$$

$$= \boxed{2.35\ Nm}$$

c) SUPPOSE COIL FLIPS FROM $\theta = 0$ TO $\theta = 180°$. HOW MUCH DOES ITS ENERGY CHANGE?

$$\Delta U = U(180) - U(0) = [-\mu B \cos 180°] - [-\mu B \cos 0°]$$

$$= -\mu B (\underbrace{\cos 180}_{-1} - \underbrace{\cos 0}_{+1}) = +2\mu B$$

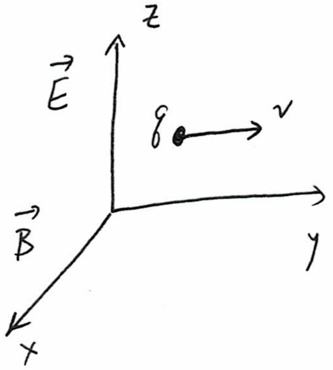
- THIS IS USED IN MRI! AN OSCILLATING $\vec{B}$ FIELD USED TO FLIP NUCLEI (WHICH HAVE MAGNETIC MOMENTS). MEASURE ENERGY ABSORBED AT DIFFERENT FREQUENCIES.

EXAMPLE: CROSSED $\vec{E}$ & $\vec{B}$ FIELDS,

A PARTICLE OF CHARGE q PASSES THROUGH CROSSED $\vec{E}$ & $\vec{B}$ FIELDS AS SHOWN.

FOR WHAT v IS PARTICLE UNDEFLECTED?

(ASSUME $q > 0$)



ELECTRIC FORCE:

$$F_E = qE \quad \text{(ALONG +z AXIS)}$$

MAGNETIC FORCE:

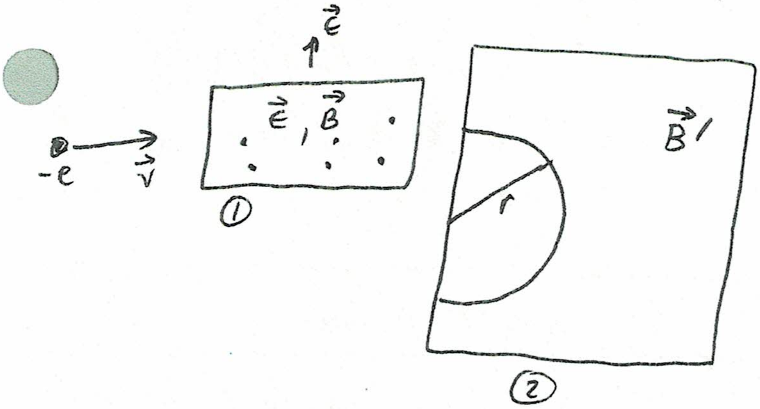
$$F_B = qvB \quad \text{(ALONG −z AXIS)}$$

SET FORCES EQUAL:

$$F_E = F_B \quad \Rightarrow \quad qE = qvB \quad \Longrightarrow \quad \boxed{v = \frac{E}{B}}$$

WILL SEE AN APPLICATION OF THIS IN NEXT EXAMPLE. ⟶

EXAMPLE – MASS SPECTROMETER ⇒ MEASURE $\frac{e}{m}$ OF ELECTRON. (OR PROTON)



REGION ①: $\vec{B}$ INTO PAGE, $\vec{E}$ UPWARD. ADJUST SIZE OF $\vec{E}$ SO PATH IS STRAIGHT. ⇒ $v = \frac{E}{B}$.

REGION ②: $\vec{B}'$ INTO PAGE

$$F = ma \quad \Rightarrow \quad e\cancel{v}B' = m\frac{v^{\cancel{2}}}{r} \quad \Rightarrow \quad eB' = \frac{mv}{r}$$

$$\frac{e}{m} = \frac{v}{B'r}$$

NOW SUBSTITUTE IN $v = \frac{E}{B}$:

$$\boxed{\frac{e}{m} = \frac{E}{BB'r}}$$

CH. 28: MAGNETIC FIELD SOURCES

LAST CHAPTER: GIVEN $\vec{B}$, FIND $\vec{F}$.

THIS CHAPTER: HOW DO YOU FIND $\vec{B}$?

TWO METHODS:

1. BIOT-SAVART LAW → ANALOG OF COULOMB'S LAW
2. AMPERE'S LAW → ANALOG OF GAUSS' LAW

RECALL IN COULOMB'S LAW, $\vec{E}$ DEPENDED ON $\vec{r}$ = DISTANCE VECTOR FROM CHARGE TO POINT. IN THE CASE OF BIOT-SAVART LAW, THERE IS A 2nd VECTOR: $\vec{s}$ = DIRECTION CURRENT FLOWS, SO $\vec{B}$ DEPENDS ON BOTH $\vec{r}$ & $\vec{s}$. SINCE THESE 2 VECTORS MUST BE COMBINED TO GET NEW VECTOR ($\vec{B}$), SHOULD EXPECT CROSS-PRODUCT TO APPEAR.

ANALOG OF CONSTANT ε_0 (PERMITTIVITY OF FREE SPACE) FOR MAGNETIC FIELDS IS:

$$\mu_0 = 4\pi \times 10^{-7} \frac{Tm}{A} = 1.26 \times 10^{-6} \frac{Tm}{A}$$

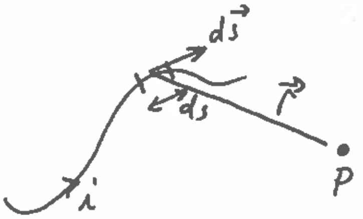
= PERMEABILITY OF FREE SPACE

NOTE THAT:

$$\mu_0 \varepsilon_0 = \left(3 \times 10^{8} \frac{m}{s}\right)^2 = (\text{SPEED OF LIGHT})^2$$

BIOT - SAVART LAW:

$$\vec{B} = \int d\vec{B}, \quad \text{WHERE:}$$



$$\boxed{dB = \frac{\mu_0}{4\pi} \cdot \frac{i\, d\vec{s} \times \hat{r}}{r^2}}$$

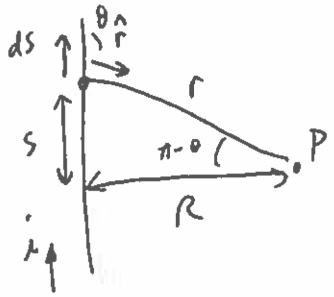
AT POINT P DUE TO SEGMENT ds

$d\vec{s}$ = SMALL LENGTH VECTOR TANGENT TO CURRENT

$\hat{r}$ = UNIT VECTOR POINTING ALONG $\vec{r}$

$\vec{r}$ = VECTOR FROM ds TO P.

LONG STRAIGHT WIRE



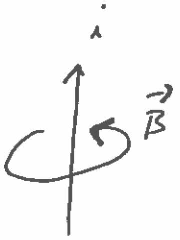
$$dB = \frac{\mu_0}{4\pi} \frac{i\, ds \sin\theta}{r^2} \quad \Bigg| \quad \begin{aligned} \sin\theta &= \sin(\pi - \theta) \\ &= R/r = \frac{R}{\sqrt{s^2+R^2}} \\ r &= \sqrt{s^2+R^2} \end{aligned}$$

$$B = \frac{\mu_0 i}{4\pi} \int_{-\infty}^{\infty} \frac{ds\, R}{(s^2+R^2)^{3/2}}$$

$$= \frac{\mu_0 i}{4\pi R} \left. \frac{s}{(s^2+R^2)^{1/2}} \right|_{-\infty}^{\infty} = \frac{\mu_0 i}{2\pi R}$$

$$\boxed{B = \frac{\mu_0 i}{2\pi R}}$$

DIRECTION

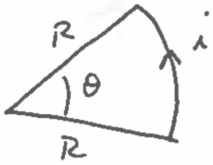


COUNTER-CLOCKWISE AS VIEWED FROM ABOVE

-OR EQUIVALENTLY-

CLOCKWISE AS VIEWED FROM BELOW

CIRCULAR ARC OF WIRE

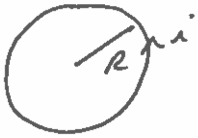


$$|d\vec{s} \times \hat{r}| = ds = R\,d\varphi$$

$$B = \frac{\mu_0}{4\pi}\int_0^\theta \frac{iR\,d\varphi}{R^2} = \frac{\mu_0 i}{4\pi R}\int_0^\theta d\varphi$$

$$\boxed{B = \frac{\mu_0 i}{4\pi R}\theta}$$ (OUT OF PAGE)

FULL CIRCULAR LOOP:
(AT CENTER)

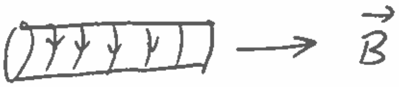


$$\xrightarrow[\theta = 2\pi]{} \quad \boxed{B = \frac{\mu_0 i}{2R}}$$

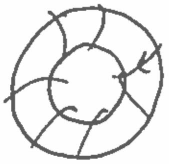
(DIRECTION: OUT OF PAGE)

SOLENOID (n LOOPS PER LENGTH):

$$\boxed{B = \mu_0 n i}$$



TOROID:



$$\boxed{B \approx \begin{cases} \frac{\mu_0 i N}{2\pi r} & \text{INSIDE} \\ 0 & \text{OUTSIDE} \end{cases}}$$

NOTE: AS $r \to \infty$, THIS GIVES RESULT FOR SOLENOID, WITH $n = \frac{N}{2\pi R}$

WILL SHOW THESE USING AMPERE'S LAW BELOW

CASES SO FAR:

LONG STRAIGHT WIRE: $B = \mu_0 i/2\pi R$
CIRCULAR LOOP : $B = \mu_0 i/2R$
SOLENOID : $B = \mu_0 n i$
TOROID : $B = \mu_0 i N/2\pi r$

(ARC OF CIRCULAR LOOP: $B = \frac{\mu_0 i}{4\pi R}\theta$)

EXAMPLE:

a) SOLENOID HAS LENGTH $L = .2M$ & CURRENT $i = 1 A$.

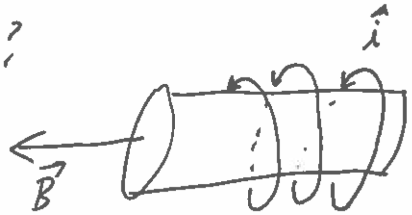
# OF TURNS: $N = 100$

FIND B AT CENTER.

$$n = \frac{N}{L} = \frac{100}{.2} = 500\ m^{-1}$$

$$B = \mu_0 n I = (4\pi \times 10^{-7})(500)(1) = \boxed{6.28 \times 10^{-4}\ T}$$

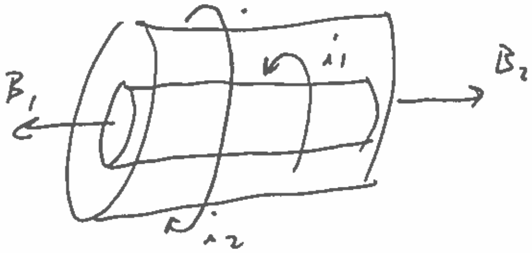
b) IF THE SOLENOID IS HORIZONTAL, WITH CURRENT GOING IN DIRECTION SHOWN, WHAT DIRECTION IS $\vec{B}$?



c) A SECOND SOLENOID IS PLACED AROUND 1ST ONE, SAME n & i, BUT WITH i IN OPPOSITE DIRECTION.

B AT CENTER NOW?

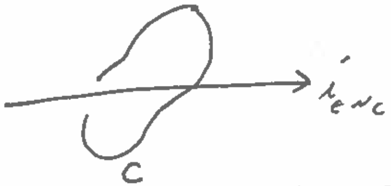
$$B_1 = -B_2 \rightarrow B = B_1 + B_2 = \boxed{0}$$



NOTE THAT B AT CENTER IS INDEPENDENT OF RADIUS OF SOLENOID.

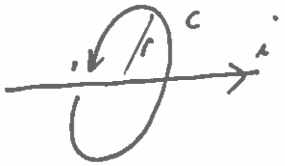
AMPERE'S LAW: MAGNETIC ANALOG OF GAUSS' LAW.

$$\boxed{\int_C \vec{B}\cdot d\vec{s} = \mu_0 i_{ENC}}$$



(i_{ENC} = CURRENT CROSSING SURFACE BOUNDED BY C)

EXAMPLE: DERIVE FORMULA FOR B OF LONG STRAIGHT WIRE USING AMPERE'S LAW.



C = CIRCLE CENTERED ON i

$i_{ENC} = i$

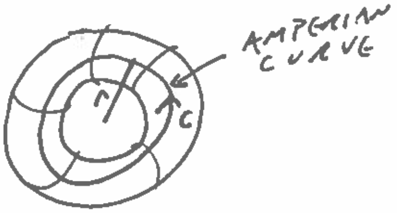
$ds = r\,d\theta$

B = CONSTANT ON C SO:

$$\int \vec{B}\cdot d\vec{s} = B\,(\underbrace{\cos 0°}_{1})\int ds = B\,r\int_0^{2\pi} d\theta = 2\pi r B$$

$$\int \vec{B}\cdot d\vec{s} = \mu_0 i_{ENC} \Rightarrow B\,2\pi r = \mu_0 i \rightarrow \boxed{B = \frac{\mu_0 i}{2\pi r}}$$

EXAMPLE: DERIVE B OF TOROID.



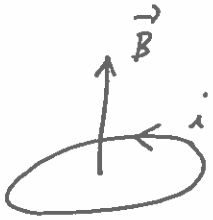
$$\int \vec{B}\cdot d\vec{s} = B\,2\pi r = \mu \underbrace{Ni}_{i_{ENC.}}$$

SO: $\boxed{B = \frac{\mu_0 N i}{2\pi r}}$

EXAMPLE: a) FIND FIELD (MAGNITUDE & DIRECTION) AT CENTER OF CURRENT LOOP OF RADIUS $r = .2m$ & CURRENT $i = 40$ A. (DIRECTION SHOWN BELOW)

$$B = \frac{\mu_0 i}{2R} = \frac{(4\pi \times 10^{-7})(40)}{2(.2)} = 4\pi \times 10^{-5} = \boxed{1.26 \times 10^{-4}\,T}$$

$(= 1.26\ G)$



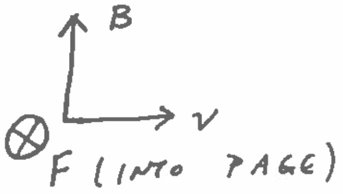
b) FIND FORCE AND ACCELERATION ON ELECTRON PASSING THROUGH ORIGIN FROM LEFT TO RIGHT AT $v = 2 \times 10^{8} \frac{m}{s}$.

$$F = |qvB \sin\theta| = (1.6 \times 10^{-19})(2 \times 10^{8})(1.26 \times 10^{-4}) \sin 90°$$

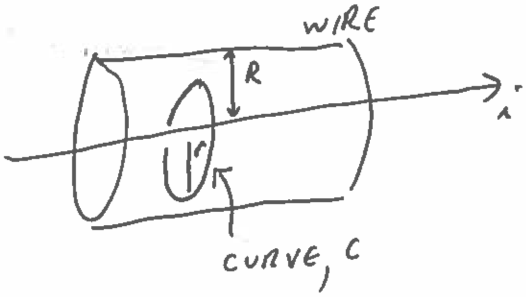
$$= 4 \times 10^{-15}\ N$$

$$a = \frac{F}{m} = \frac{4 \times 10^{-15} N}{9.11 \times 10^{-31} kg} = \boxed{4.4 \times 10^{15} \frac{m}{s^2}}$$

DIRECTION:



EXAMPLE: CONSIDER LONG, STRAIGHT WIRE OF RADIUS R, CARRYING CURRENT i. DERIVE EXPRESSION FOR B FIELD AT DISTANCE r FROM CENTER INSIDE WIRE $(r<R)$. (ASSUME UNIFORM CURRENT)



CURRENT DENSITY:

$$J = \frac{i}{A_{TOTAL}} = \frac{i}{\pi R^2}$$

CURRENT ENCLOSED BY C:

$$i_{ENC} = J A_{ENC} = \frac{i}{\pi R^2} \cdot \pi r^2$$

$$= \frac{r^2}{R^2} i$$

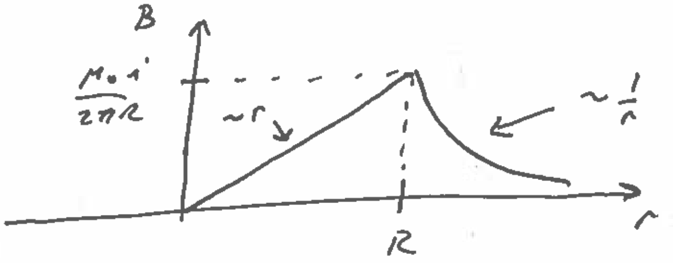
SO:

$$\int \vec{B} \cdot d\vec{s} = B(2\pi r) = \mu_0 i_{ENC}$$

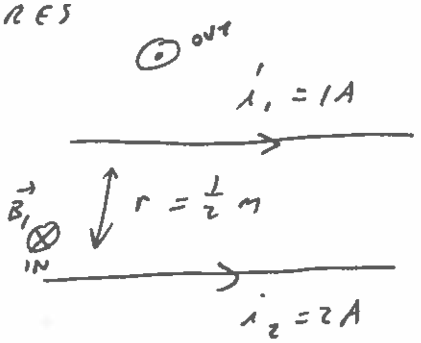
$$B = \frac{\mu_0 i_{ENC}}{2\pi r} = \frac{\mu_0}{2\pi r} \cdot \frac{r^2}{R^2} \cdot i = \frac{\mu_0 i r}{2\pi R^2}$$

$$\boxed{B = \frac{\mu_0 i r}{2\pi R^2}}$$ (INSIDE WIRE)

OUTSIDE: $B = \frac{\mu_0 i}{2\pi r}$



EXAMPLE: FORCE BETWEEN 2 WIRES



$l_1 = l_2 = 3m$

B FROM FIRST WIRE:

$$B_1 = \frac{\mu_0 i_1}{2\pi r} \quad \text{(INTO PAGE)}$$

FORCE ON 2nd WIRE:

$$F_2 = i_2 l_2 B_1 \sin\theta \quad (\theta = 90°)$$

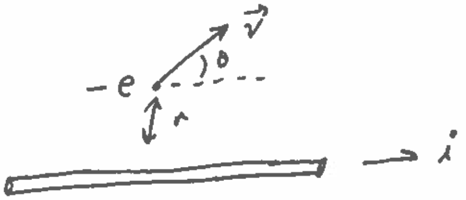
$$= \frac{\mu_0 i_1 i_2 l_2}{2\pi r}\sin\theta$$

$$= \frac{(4\pi\times10^{-7})(1)(2)(3)}{2\pi(\frac{1}{2})} \cdot \underbrace{\sin 90°}_{1} = \boxed{2.4\times10^{-6} N} \quad \text{(UPWARD)}$$

FORCE ON 1ST WIRE: EQUAL & OPPOSITE.

NOTE: CURRENTS IN SAME DIRECTION → ATTRACT
CURRENTS IN OPPOSITE DIRECTIONS → REPEL

EXAMPLE:



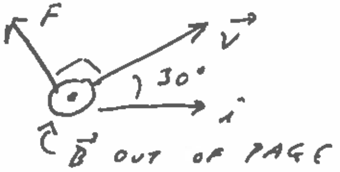
$i = 20A$
$v = 5\times10^7 \frac{m}{s}$
$\theta = 30°$
$r = .1m$

a) FIND MAGNETIC FORCE ON ELECTRON $r = .1m$ AWAY.

i) FIND $\vec{B}$: $B = \frac{\mu_0 i}{2\pi r} = \frac{(4\pi\times10^{-7})(20)}{2\pi(.1)} = 4\times10^{-5}\,T$ (OUT OF PAGE)

ii) FIND $\vec{F}$: $F = |qvB\sin\theta| = (1.6\times10^{-19})(5\times10^{7})(4\times10^{-5})\sin 90°$

$= \boxed{3.2\times10^{-16} N}$ DIRECTION:



b) FIND ACCELERATION

$$a = \frac{F}{m} = \frac{3.2\times10^{-16}N}{9.1\times10^{-31}kg} = \boxed{7.04\times10^{14}\frac{m}{s^2}}$$

(IN DIRECTION OF $\vec{F}$)

MAGNETISM OF MATTER

$\vec{\mu}$ = MAGNETIC MOMENT PER MOLECULE

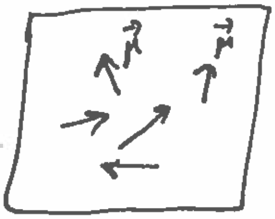
$\vec{M}$ = MAGNETIZATION OF MATERIAL

$= \dfrac{\Sigma \vec{\mu}}{V}$, WHERE SUM IS OVER ALL MOLECULES IN VOLUME V

THERE ARE THREE TYPES OF MAGNETIC BEHAVIOR A MATERIAL CAN HAVE:

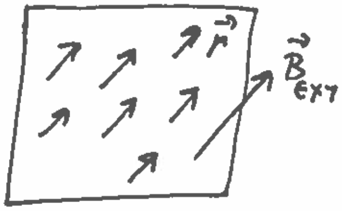
I. PARAMAGNETISM

- PERMANENT $\vec{\mu}$'S ARE RANDOMLY ORIENTED & TEND TO CANCEL $\Longrightarrow$ $\vec{M} = 0$, WHEN NO EXTERNAL FIELD IS PRESENT
- WHEN EXTERNAL FIELD $\vec{B}_{EXT}$ IS TURNED ON, $\vec{\mu}$'S TEND TO ALIGN WITH $\vec{B}_{EXT}$, SO $\vec{M} \neq 0$ AS LONG AS $\vec{B}_{EXT}$ IS PRESENT.
- THE $\vec{\mu}$'S PRODUCE THEIR OWN INDUCED FIELD, $\vec{B}_{IND}$. $\vec{B}_{IND}$ IS $\parallel$ TO $\vec{B}_{EXT}$ $\longrightarrow$ TOTAL FIELD $(\vec{B}_{EXT} + \vec{B}_{IND})$ INSIDE MATERIAL IS STRONGER THAN FIELD OUTSIDE $(\vec{B}_{EXT})$
- MATTER FEELS NET FORCE $\vec{F}$ PULLING IT TOWARD REGIONS OF STRONGER $\vec{B}_{EXT}$



$B_{EXT} = 0$

$\Longrightarrow$



$B_{EXT} \neq 0$

$\Longrightarrow$

II. DIAMAGNETISM

- NOW $\vec{\mu}$'S TEND TO ALIGN OPPOSITE TO $\vec{B}_{EXT}$. SO $\vec{B}_{TOTAL} = (\vec{B}_{EXT} + \vec{B}_{IND})$ IS SMALLER THAN $\vec{B}_{EXT}$.
- NO PERMANENT $\vec{M}$ WHEN $\vec{B}_{EXT}$ NOT PRESENT
- MATERIAL IS PULLED TOWARD REGIONS OF SMALLER $\vec{B}_{EXT}$.

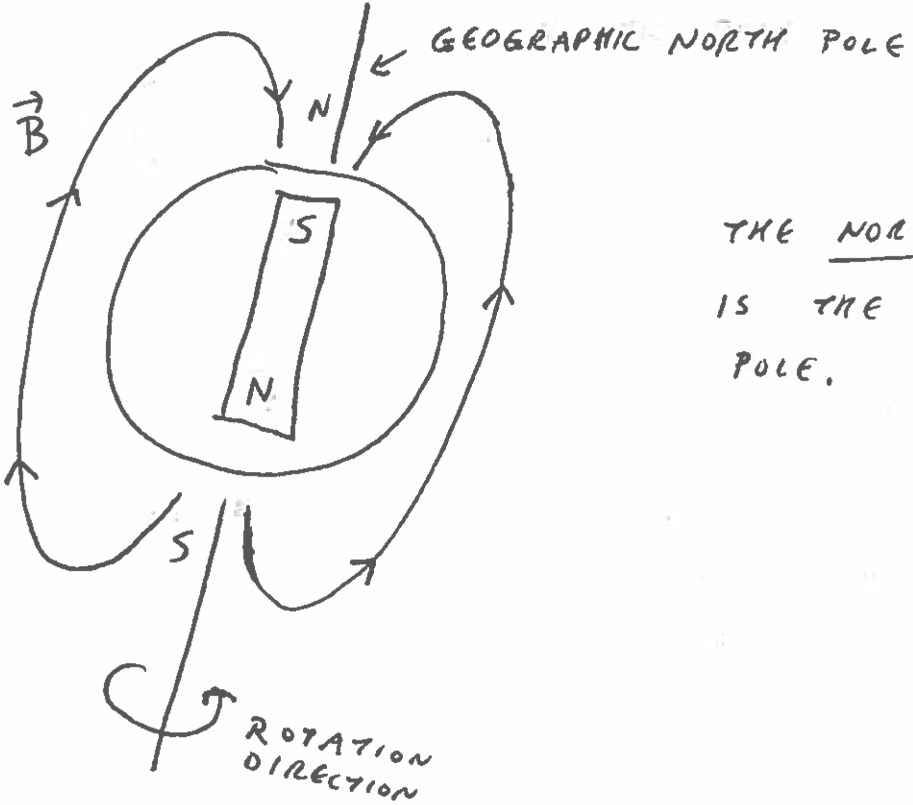
III. FERROMAGNETISM (IRON, NICKEL, COBALT)

- AT HIGH TEMPERATURES (ABOVE T_c = CURIE TEMP.), $\vec{\mu}$'S CAN ALIGN WITH $\vec{B}_{EXT}$.
- IF TEMP. DROPS, THE $\vec{\mu}$'S WILL "FREEZE" IN PLACE ⟹ MAGNETIZATION M PERSISTS EVEN AFTER $\vec{B}_{EXT}$ IS REMOVED. ⟹ $\boxed{\text{CAN HAVE PERMANENT MAGNETIZATION}}$
- MATERIAL IS ATTRACTED TO STRONGER $\vec{B}_{EXT}$.

IN EACH CASE, MATERIAL IS ATTRACTED TO REGION WHERE ENERGY IS LOWEST.

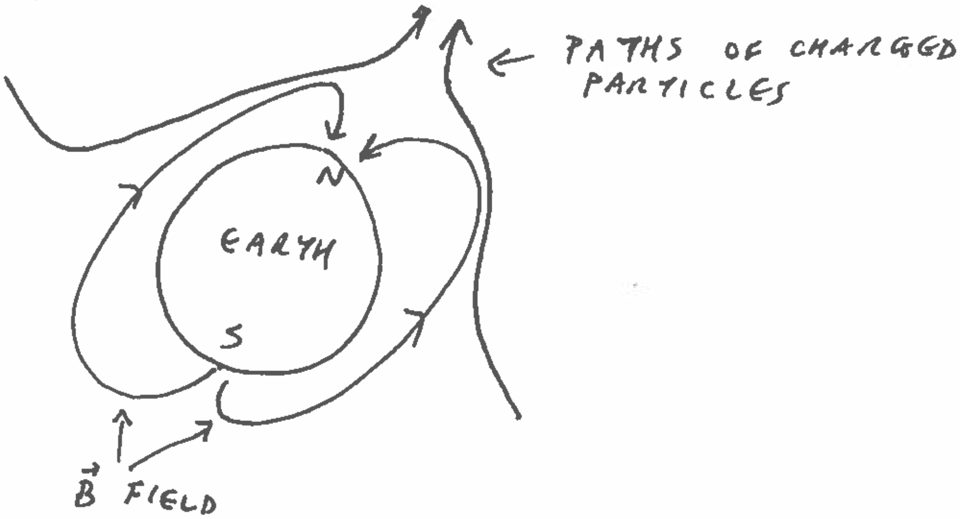
$$U = -(\Sigma\vec{\mu})\cdot\vec{B} = -V\vec{M}\cdot\vec{B}$$

THE EARTH ACTS LIKE A PERMANENT FERROMAGNET.


GEOGRAPHIC NORTH POLE
$\vec{B}$
N
S
N
S
ROTATION DIRECTION

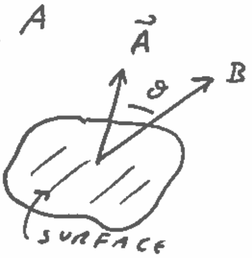
THE NORTH GEOGRAPHIC POLE IS THE SOUTH MAGNETIC POLE.

THE EARTH'S MAGNETIC FIELD IS IMPORTANT FOR LIFE: ELECTRICALLY CHARGED COSMIC RAYS CONSTANTLY BOMBARD THE ATMOSPHERE. BUT MOST OF THE GET SWEPT ALONG THE MAGNETIC FIELD LINES AND EJECTED AT THE POLES. AS A RESULT THEY DON'T REACH THE EARTH'S SURFACE TO DAMAGE DNA OF LIVING CREATURES.


PATHS OF CHARGED PARTICLES
N
EARTH
S
$\vec{B}$ FIELD

CH. 29: INDUCTION & FARADAY'S LAW

MAGNETIC FLUX: $\Phi_B = \vec{B}\cdot\vec{A} = BA\cos\theta = B_\perp A$



$\vec{A}$ = VECTOR ⊥ TO SURFACE.

(MAGNITUDE OF $\vec{A}$) = AREA

UNITS OF FLUX:

WEBERS: $1\ Wb = 1\ Tm^2$

NOTE THAT: $1\ \frac{Wb}{s} = 1\ \frac{Tm^2}{s} = 1\ V$

SUPPOSE FLUX Φ_B THROUGH LOOP OF WIRE OF AREA A IS CHANGING OVER TIME. THE CHANGING FLUX PRODUCES AN INDUCED EMF $\mathcal{E}$ & AN INDUCED CURRENT i IN THE LOOP:

FARADAY'S LAW: $$\boxed{\mathcal{E} = -N\frac{d\Phi_B}{dt}}$$ (N = # OF LOOPS)

MINUS ⇒ REMINDER OF LENS' LAW

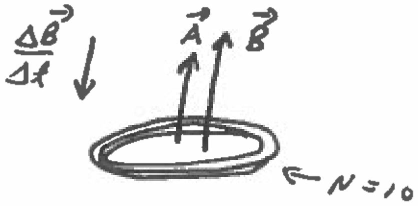
LENZ'S LAW: DIRECTION OF INDUCED $\mathcal{E}$ & I ARE SUCH THAT THEY OPPOSE CHANGE IN FLUX.

NOTES:

1. INDUCED CURRENT IS GIVEN BY $i = \mathcal{E}/R$
2. $\frac{d\Phi_B}{dt}$ CAN BE RESULT OF:
   - i) CHANGING $\vec{B}$
   - ii) CHANGING A
   - iii) CHANGING θ

EXAMPLES

① A SOLENOID WITH 10 LOOPS & CROSS-SECTION $A = .4\,m^2$ HAS A RESISTANCE $R = 5\,\Omega$. A $\vec{B}$ FIELD POINTS PARALLEL TO THE AXIS. B IS DECREASING AT RATE 3T/s (SO $\frac{\Delta B}{\Delta t} = -3\,\frac{T}{s}$).


$\frac{\Delta \vec{B}}{\Delta t}$
$\vec{A}$
$\vec{B}$
$\leftarrow N=10$

a) FIND SIZE OF INDUCED $\mathcal{E}$ & I.

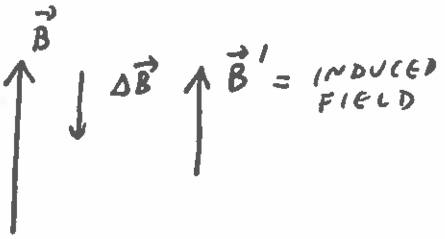
$$|\mathcal{E}| = \left| N \frac{\Delta \Phi_B}{\Delta t} \right| = N \left| \frac{A \cos\theta \, \Delta B}{\Delta t} \right|$$

$$= (10)\,(.4\,m^2)\,(\cos 0^\circ)\left(3\,\frac{T}{s}\right)$$

$$= 12\,\frac{T m^2}{s} = \boxed{12\ V}$$

$$I = \frac{|\mathcal{E}|}{R} = \frac{12\ V}{5\,\Omega} = \boxed{2.4\ A}$$

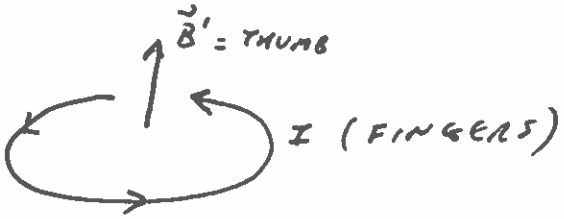
b) WHAT IS THE DIRECTION OF THE CURRENT?


$\vec{B}$
$\Delta \vec{B}$
$\vec{B}'$ = INDUCED FIELD

$\vec{B}'$, THE FIELD INDUCED BY THE NEW CURRENT OPPOSES CHANGE IN ORIGINAL FIELD, $\Delta \vec{B}$ BY LENZ'S LAW.

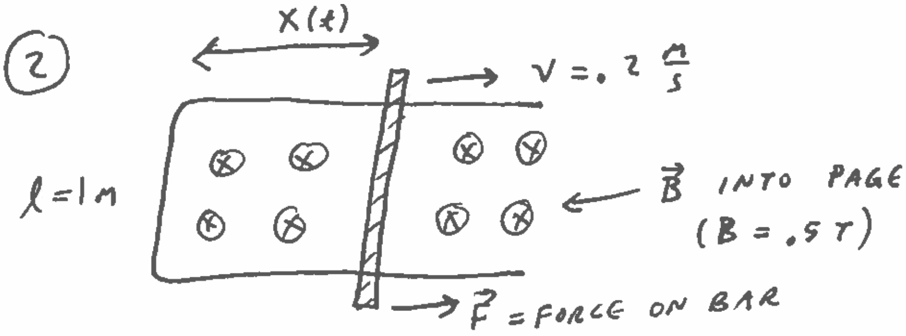
SO $\vec{B}'$ IS UP. $\xrightarrow{\hspace{2em}}$ RIGHT-HAND RULE

$\boxed{\text{I IS COUNTER-CLOCKISE AS VIEWED FROM ABOVE}}$


$\vec{B}'$ = THUMB
I (FINGERS)

VOLTAGE IN LOOP INCREASES AS YOU GO IN DIRECTION OF I.

②



RESISTANCE OF CLOSED LOOP (AT SOME PARTICULAR TIME):

$R = 10\ \Omega$

A METAL BAR LAYS ON TOP OF A U-SHAPED METAL WIRE. IT IS PULLED TO THE RIGHT, SLIDING OVER THE WIRE AT CONSTANT SPEED $v = .2\ \frac{m}{s}$.

MAGNETIC FIELD $\vec{B}$ POINTS INTO PAGE. ($\vec{B}$ IS UNIFORM)

a) FIND $\mathcal{E}$ & I

$$\mathcal{E} = N\left|\frac{\Delta\Phi_B}{\Delta t}\right| = NB\cos 0^\circ\left|\frac{dA}{dt}\right| = NB\ell\,\frac{\Delta x}{\Delta t} = NB\ell v$$

$$= (1)(.5)(1)(.2) = \boxed{.10\ V}$$

$$I = \frac{|\mathcal{E}|}{R} = \frac{.1\ V}{10\ \Omega} = .01\ A = \boxed{10\ mA}$$

AT THIS MOMENT. OVER TIME, R WILL INCREASE, SO I WILL DECREASE.

DIRECTION OF $I \Rightarrow$ $\boxed{\text{COUNTER-CLOCKWISE}}$

(Φ INCREASING $\rightarrow$ $\vec{B}'_{IND}$ OUT OF PAGE TO WEAKEN $\vec{B}_{TOTAL}$ $\xrightarrow[\text{RIGHT HND. RULE}]{}$ COUNTER-CLOCKWISE CURRENT)

b) FIND FORCE REQUIRED TO KEEP v CONSTANT.

$$F = I\ell B\sin\theta = (.01)(1)(.5)\sin 90 = \boxed{5\times10^{-3}\ N}$$

c) FIND POWER NEEDED.

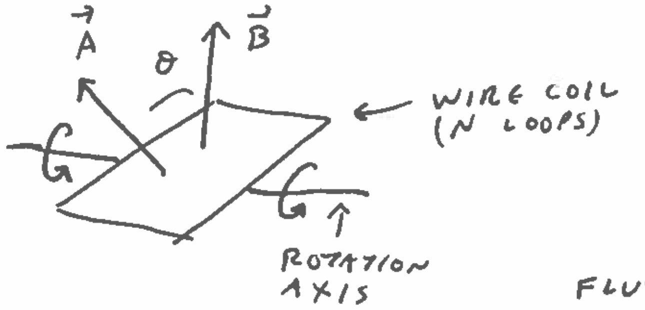
$$P_{IN} = \frac{W}{\Delta t} = \frac{F\Delta x}{\Delta t} = Fv = (5\times10^{-3})(.2) = \boxed{10^{-3}\ W}$$

POWER PUT IN BY FORCE AS WORK.

THE POWER IS LOST AGAIN THROUGH RESISTANCE!

$$P_{OUT} = I^2R = (.01\ A)^2(10\ \Omega) = 10^{-3}\ W \rightarrow$$ ENERGY STAYS CONSTANT

③ ELECTRIC GENERATOR: MECHANICAL WORK TURNED INTO A VOLTAGE


$\vec{A}$
θ
$\vec{B}$
WIRE COIL
(N LOOPS)
ROTATION AXIS

DIRECTION OF $\vec{B}$ IS FIXED. LOOP ROTATES AT CONSTANT RATE:

$$\theta = \omega t$$

FLUX:

$$\Phi_B = BA \cos \omega t$$

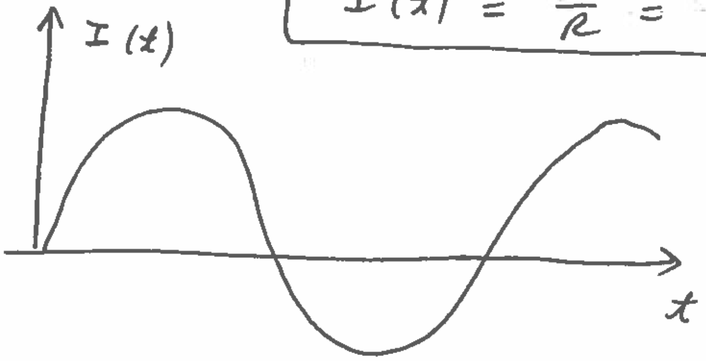
EMF:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} N = +NBA\omega \sin \omega t$$

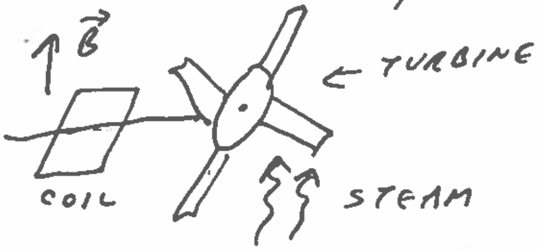
MUST USE DERIVATIVE (NOT $\frac{\Delta \Phi_B}{\Delta t}$), SINCE Φ_B IS NOT LINEAR IN t.

THE VOLTAGE AND CURRENT ARE ALTERNATING CURRENT (AC):

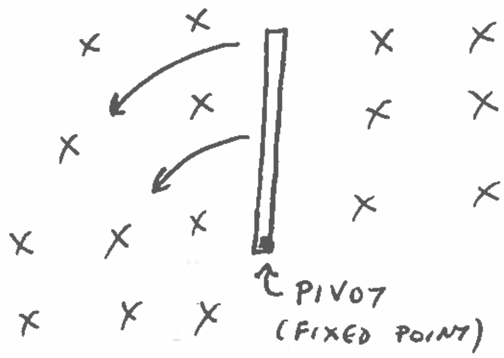
$$\boxed{\begin{aligned} \mathcal{E}(t) &= NBA\omega \cdot \sin \omega t \\ I(t) &= \frac{\mathcal{E}}{R} = \frac{NBA\omega}{R} \sin \omega t \end{aligned}}$$



THE ROTATION MAY BE PROVIDED BY WATERFALL, TIDES, RISING STEAM, ETC.


$\vec{B}$
TURBINE
COIL
STEAM

④

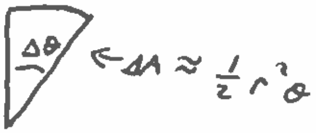


BAR OF LENGTH $r = .1\,m$, PIVOTED AT ONE END ROTATES AT CONSTANT ANGULAR SPEED $\omega = 8\pi\ \frac{RAD}{S}$. IT PASSES THROUGH UNIFORM FIELD $B = 2T$.

WHAT IS THE POTENTIAL DIFFERENCE BETWEEN THE ENDS OF THE BAR?

AREA SWEPT OUT PER TIME:

$$\frac{\Delta A}{\Delta t} = \frac{\left(\frac{1}{2} r^2 \Delta\theta\right)}{\Delta t} = \frac{1}{2} r^2 \omega$$



$$|\mathcal{E}| = \left|N \frac{\Delta\Phi_B}{\Delta t}\right| = (\underset{N}{1}) \frac{\Delta\Phi_B}{\Delta t} = \frac{B \cos 90^\circ \Delta A}{\Delta t} = B\underbrace{(\cos 90)}_{1}\left(\frac{1}{2} r^2 \omega\right)$$

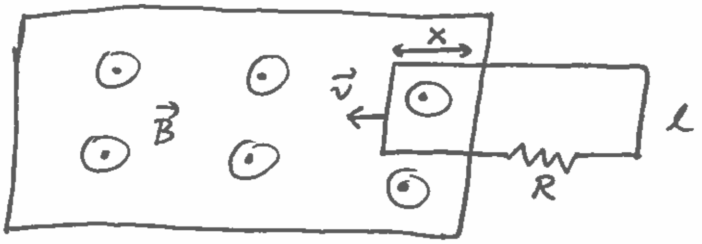
$$= (2T)\left(\frac{1}{2}\right)(.1m)^2\left(8\pi \frac{RAD}{S}\right) = .08\pi$$

$$= \boxed{.251\,V}$$

NOTE THAT THERE IS NO CURRENT (THERE IS NO CLOSED LOOP FOR IT TO FLOW THROUGH).

BUT THE TWO ENDS OF THE BAR CAN SERVE AS A BATTERY.

⑤



$V = .5 \frac{m}{s}$

$B = 3\,T$ (OUT)

$\ell = .1\,m$

$R = 2\,\Omega$

WIRE LOOP OF RESISTANCE R ENTERS REGION OF UNIFORM $\vec{B}$ (OUT OF PAGE), SPEED v OF LOOP IS CONSTANT. X = DISTANCE TO WHICH LOOP HAS PENETRATED INTO FIELD.

a) INDUCED EMF ?

$$|\mathcal{E}| = N\left|\frac{d\Phi_B}{dt}\right| = NB\frac{dA}{dt} = NB\ell\frac{dx}{dt}$$

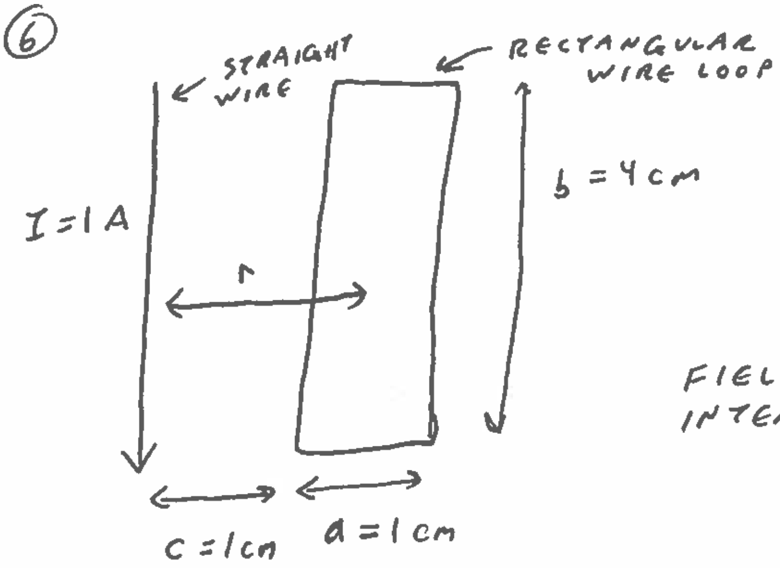
$$= NB\ell v = (1)(3)(.1)(.5) = \boxed{.15\ V}$$

b) CURRENT ?

$$I = \frac{|\mathcal{E}|}{R} = \frac{.15\ V}{2\,\Omega} = \boxed{.075\ A} = 75\ mA$$

DIRECTION: $\boxed{\text{CLOCKWISE}}$

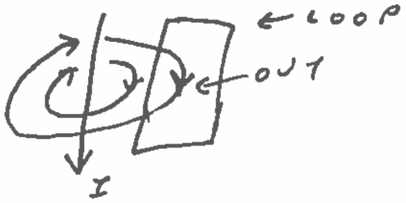
⑥


STRAIGHT WIRE
RECTANGULAR WIRE LOOP
b = 4 cm
I = 1 A
r
c = 1 cm
a = 1 cm

← THE STRAIGHT WIRE CARRIES CONSTANT CURRENT $I = 1\,A$.

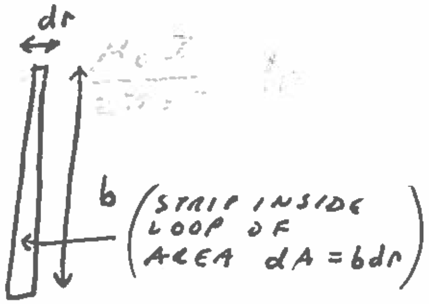
WHAT IS THE MAGNETIC FLUX THROUGH THE LOOP?

FIELD IS OUT OF PAGE IN THE INTERIOR OF THE LOOP:



AT DISTANCE r FROM WIRE:

$$B = \frac{\mu_0 I}{2\pi r}$$


dr
b
(STRIP INSIDE LOOP OF AREA dA = b dr)

FLUX:

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = \int_c^{c+a} \left(\frac{\mu_0 I}{2\pi r}\right)(b\,dr)$$

$$= \frac{\mu_0 I b}{2\pi} \int_c^{c+a} \frac{dr}{r} = \frac{\mu_0 I b}{2\pi}\left(\ln(c+a) - \ln c\right)$$

$$= \boxed{\frac{\mu_0 I b}{2\pi} \ln \frac{c+a}{c}}$$

NOTE THAT INTEGRATION IS NECESSARY (CAN'T USE PRODUCT $\Phi_B = \vec{B} \cdot \vec{A}$) BECAUSE $\vec{B}$ IS NOT CONSTANT OVER THE AREA OF THE LOOP

EXAMPLE CONTINUED →

b) SUPPOSE NOW THE CURRENT STARTS INCREASING AT A CONSTANT RATE: $\frac{dI}{dt} = .5 \frac{A}{s}$.

FIND $\mathcal{E}$.

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}\Big(\underbrace{\frac{\mu_0 I b}{2\pi} \ln \frac{c+a}{c}}_{\text{FROM PART a}}\Big) = \Big(\frac{\mu_0 b}{2\pi} \ln \frac{c+a}{c}\Big)\frac{dI}{dt}$$

$$= \frac{(4\pi \times 10^{-7})(.04\,m)}{2\pi} \ln\Big(\frac{.01+.01}{.01}\Big)$$

$$= 2.75 \times 10^{-9}\ \frac{Wb}{s}$$

$$\mathcal{E} = \frac{d\Phi_B}{dt} \cdot \underset{\uparrow N=1}{N} = \boxed{2.75 \times 10^{-9}\ V}$$

c) CURRENT INDUCED IN LOOP? (ASSUME $R = 3\ \Omega$)

$$I = \frac{|\mathcal{E}|}{R} = \frac{2.75 \times 10^{-9}\ V}{3\ \Omega} = \boxed{9.2 \times 10^{-10}\ A}$$

DIRECTION:

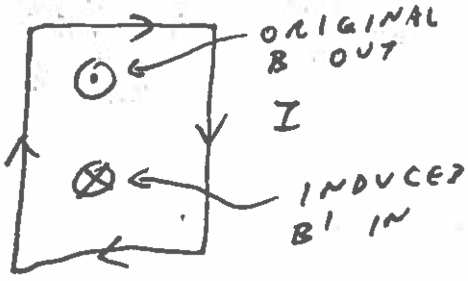
I INCREASING $\Rightarrow$ B INCREASING OUT OF PAGE

$\Rightarrow$ FLUX INCREASING

SO LENZ SAYS:

CURRENT IN LOOP WANTS TO CANCEL INCREASE BY PRODUCING NEW, INDUCED $\vec{B}'$ OPPOSED TO ORIGINAL $\vec{B}$.

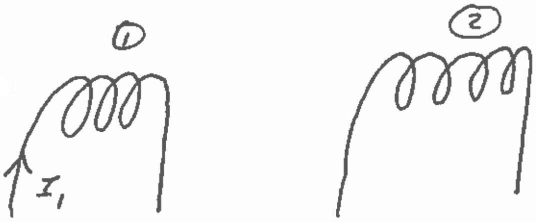
$\vec{B}'$ INTO PAGE $\xrightarrow[\text{RIGHT HAND RULE}]{}$ $\boxed{\text{I IS CLOCKWISE}}$



CH. 30: INDUCTANCE & AC CIRCUITS

INDUCTANCE

GIVEN 2 COILS:



CHANGING THE CURRENT IN ① WILL INDUCE CURRENT IN ②, BY FARADAYS LAW. THE INDUCED VOLTAGE IN ② WILL BE PROPORTIONAL TO RATE OF CHANGE OF I_1:

$$\mathcal{E}_2 = -M \frac{\Delta I_1}{\Delta t} = -\left(\frac{N_2 \Delta \Phi_{B_2}}{\Delta I_1}\right) \frac{\Delta I_1}{\Delta t} \qquad \boxed{M = \frac{N_2 \Delta \Phi_2}{\Delta I_1}}$$

THE PROPORTIONALITY CONSTANT M IS THE MUTUAL INDUCTANCE

(UNITS OF INDUCTANCE = HENRYS, $1\,H = 1\,\Omega \cdot s$)

M WILL DEPEND ON N_1, N_2, LENGTHS OF COILS, DISTANCE BETWEEN THEM, ETC.

MINUS SIGN: $\mathcal{E}$ OPPOSES CHANGE IN I (LENZ'S LAW AGAIN)

SELF-INDUCTANCE: $$\mathcal{E} = -L \frac{\Delta I}{\Delta t}$$

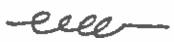
$$\boxed{L = \frac{\Delta \Phi_2}{\Delta I} N}$$

INDUCTORS RESIST LARGE CURRENT FLOWS, THE ENERGY REMOVED FROM CURRENT IS PUT INTO MAGNETIC FIELD.

$$\text{ENERGY} = \tfrac{1}{2} L I^2 \qquad \text{(INDUCTOR ONLY)}$$

$$\text{ENERGY DENSITY} = \frac{1}{2\mu_0} B^2 \qquad \text{(GENERAL)}$$

COMPARE TO $\frac{\varepsilon_0}{2} \in^2$

SYMBOL FOR AN INDUCTOR IN AN ELECTRIC CIRCUIT: 

EXAMPLES

① FIND (SELF) INDUCTANCE OF A SOLENOID.

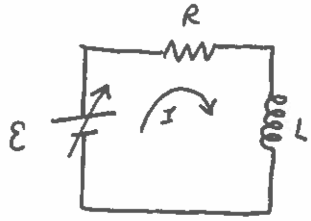
FLUX PER LOOP: $\Phi_B = BA\cos 0^\circ = BA = \mu_0 n I A$ (SINCE $B = \mu_0 n I$,

WRITE $n = \frac{N}{\ell} \Rightarrow \Phi_B = \frac{\mu_0 N I A}{\ell}$

BUT: $L = N\frac{\Delta\Phi_B}{\Delta I} = N\left(\frac{\mu_0 N A}{\ell}\right)\frac{\Delta I}{\Delta I} = \frac{\mu_0 N^2 A}{\ell}$

$$\boxed{L = \frac{\mu_0 N^2 A}{\ell}}$$

②



$R = .10\ \Omega$ $L = 2\times10^{-2}\,H$

$\mathcal{E}$ IS DECREASING ⟹ I IS DECREASING

a) AT A GIVEN MOMENT, $I = 2\,A$ & $\left|\frac{\Delta I}{\Delta t}\right| = .3\ \frac{A}{s}$.

FIND $\mathcal{E}$, V_R, & V_L AT THIS MOMENT.

$V_R = RI = (.10)(2) = \boxed{.20\,V}$ (DECREASING AS WE GO IN DIRECTION OF CURRENT)

$V_L = \underset{\uparrow \text{ ABS. VALUE}}{L\frac{\Delta I}{\Delta t}} = (2\times10^{-2})(.3) = \boxed{6\times10^{-3}\,V}$ (INCREASING)

$\mathcal{E} = V_R - V_L = .2 - .006$

$= \boxed{.194\,V}$ (NOTE: $\mathcal{E} \neq IR$)

b) HOW MUCH ENERGY IS STORED IN INDUCTOR?

$\frac{1}{2}LI^2 = \frac{1}{2}(2\times10^{-2})(2)^2 = \boxed{4\times10^{-2}\,J}$

CONTINUED

c) FIND THE CURRENT, $I(t)$.

KIRCHOFF: $\mathcal{E} - IR - L\frac{dI}{dt} = 0$

$$L\frac{dI}{dt} = \mathcal{E} - IR$$

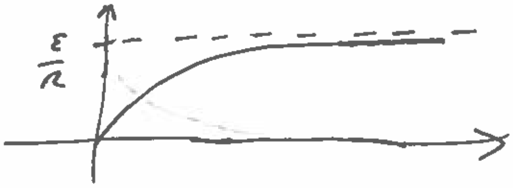
$$\int_0^I \frac{dI}{\mathcal{E}-IR} = \int \frac{1}{L}\,dt = \frac{t}{L}$$

CHANGE VARIABLE: $y = \mathcal{E} - IR \rightarrow dy = -R\,dI$

$$\frac{t}{L} = \int_{\mathcal{E}}^{\mathcal{E}-IR} -\frac{dy}{Ry} = -\frac{1}{R}\ln y\Big|_{\mathcal{E}}^{\mathcal{E}-IR}$$

$$\frac{t}{L} = -\frac{1}{R}\ln\frac{\mathcal{E}-IR}{\mathcal{E}} \rightarrow \mathcal{E} - IR = \mathcal{E}e^{-\frac{Rt}{L}}$$

$$\boxed{I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{Rt}{L}}\right)}$$



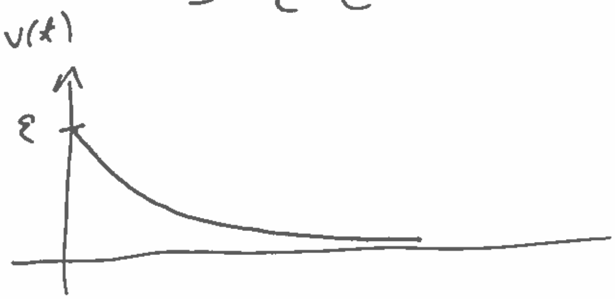
THE RATIO $\frac{L}{R}$ IS CALLED THE TIME CONSTANT!

$$\boxed{\tau = \frac{L}{R}}$$

VOLTAGE ACROSS L AT TIME t:

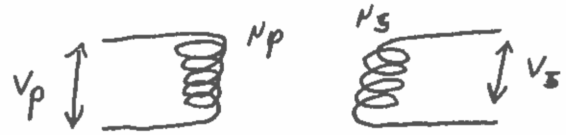
$$V(t) = -L\frac{dI}{dt} = -\frac{L\mathcal{E}}{R}\left(-\frac{R}{L}\right)e^{-\frac{Rt}{L}}$$

$$= \mathcal{E}e^{-Rt/L}$$



NOTE THE SIMILARITIES TO THE CHARGE $Q(t)$ ON A CAPACITOR.

EXAMPLE: TRANSFORMERS



APPLY ALTERNATING CURRENT TO THE 1ST COIL (PRIMARY), V_p = AMP. OF PRIM. VOLTAGE.

WHAT IS V_s, THE AMPLITUDE OF INDUCED VOLTAGE IN SECONDARY COIL?

$$V_p = N_p \frac{\Delta \Phi}{\Delta t} \qquad V_s = N_s \frac{\Delta \Phi}{\Delta t}$$

EQUAL CHANGES IN FLUX

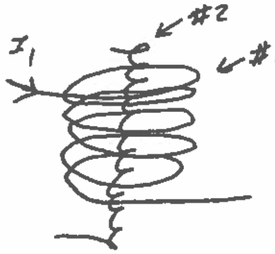
$$\text{SO: } \frac{V_p}{N_p} = \frac{V_s}{N_s} \Rightarrow \boxed{V_s = \frac{N_s}{N_p} V_p}$$

SO, CAN USE TRANSFORMER TO MAGNIFY VOLTAGE BE MAKING [illegible] $N_s > N_p$. THIS IS DONE AT EXPENSE OF CURRENT.

$$\text{POWER IN = POWER OUT} \Rightarrow I_p V_p = I_s V_s$$

$$I_s = \left(\frac{V_p}{V_s}\right) I_p = \frac{N_p}{N_s} I_p \Rightarrow \boxed{I_s = \frac{V_p}{V_s} I_p}$$

EXAMPLE: 2 COILS OF SAME LENGTH, SAME NUMBER OF LOOPS. (GIVEN: N & A_2)



FIND M.

$$B_1 = \mu_0 N_1 I_1$$

$$\frac{\Delta \Phi_2}{\Delta t} = \frac{\Delta (B_1 A_2)}{\Delta t} = A_2 \mu_0 N \frac{\Delta I_1}{\Delta t}$$

$$M = N \frac{\Delta \Phi_2}{\Delta I_1} = N (A_2 \mu_0 N) = \boxed{\mu_0 N^2 A_2}$$

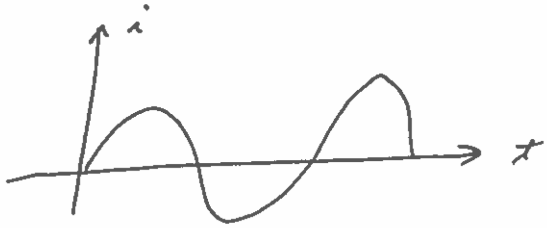
INDUCTORS IN CIRCUITS:

INDUCTORS OPPOSE CHANGES IN CURRENTS: THEY CREATE VOLTAGES OF FORM:

$$V = -L\frac{di}{dt}$$

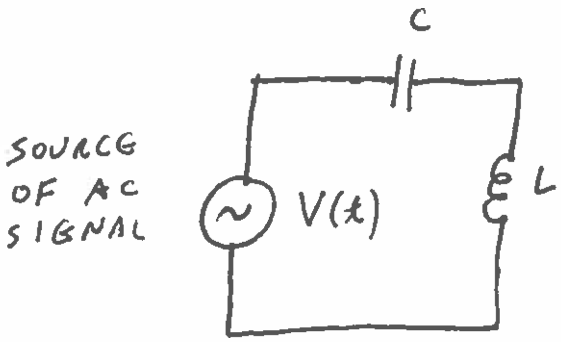
THEY THUS REGULATE HOW FAST CURRENTS CAN CHANGE IN CIRCUITS.

TOGETHER WITH CAPACITORS, INDUCTORS CAN CAUSE CIRCUITS TO OSCILLATE:



APPLICATIONS:

- RADIO TRANSMITTERS & RECIEVERS.
- CIRCUIT TO PRODUCE SOUNDS OF GIVEN FREQUENCY. (SYNTHESIZERS)



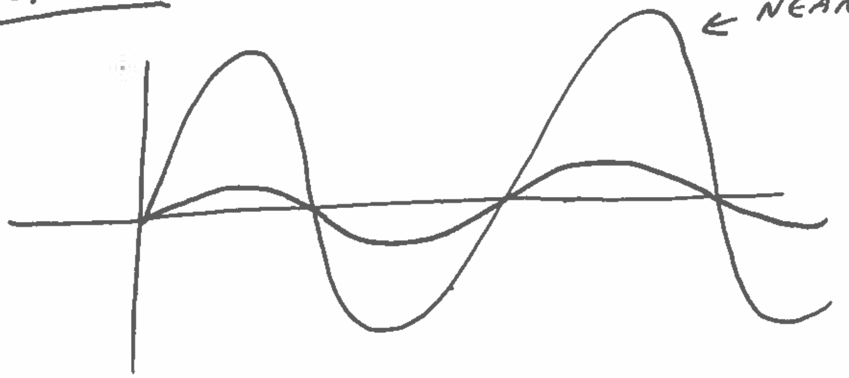
INPUT SIGNAL:

$$V(t) = V_0 \sin \omega t$$

THE CIRCUIT PASSES SOME FREQUENCIES & BLOCKS OTHERS →

LC CIRCUIT:

RESONANCE OCCURS AT $\omega_0 = 2\pi f = \frac{1}{\sqrt{LC}}$.

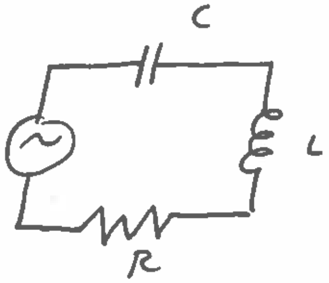


← NEAR RESONANCE! $\omega \approx \omega_0$

← FAR FROM RESONANCE: $\omega \neq \omega_0$

RLC CIRCUIT: NOW ADD RESISTOR.

TWO EFFECTS OCCUR:



1. RESONANT FREQUENCY IS SHIFTED:

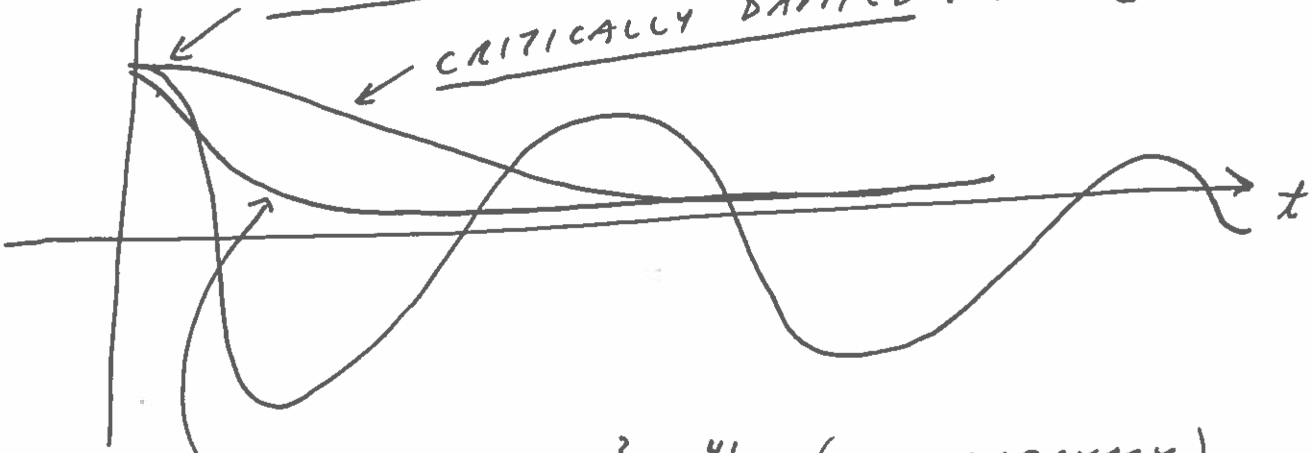
$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

2. WAVE NOW DECAYS OVER TIME (UNLESS MORE ENERGY IS PUMPED IN TO REPLACE WHAT IS LOST).

THREE CASES

UNDERDAMPED: $R^2 < \frac{4L}{C}$ (ω_0 REAL)

CRITICALLY DAMPED: $R^2 = \frac{4L}{C}$ ($\omega_0 = 0$)



OVERDAMPED: $R^2 > \frac{4L}{C}$ (ω_0 IMAGINARY)
(EXPONENTIAL DECAY)

Improving Today's Light TECHNOLOGY

Batteries not included... or needed

The Forever Flashlight uses the Faraday Principle of Electromagnetic Energy to eliminate the need for replacement bulbs or batteries.

- Bright Blue LED
- Never needs batteries
- Never needs bulbs
- Floats in water

Just shake to recharge!

As the magnet passes through the metal coils, energy is created.

It's a dark and stormy night, and right in the middle of your favorite television program, the power suddenly cuts off. You go to the kitchen, and search through the cabinets for the flashlight you've placed there for just such an occasion. You finally find it, and start to turn it on when the inevitable strikes: the batteries are dead. What if you could know without a doubt that every time you picked up a flashlight, it would work, providing you with the light you need? Now you can, thanks to the Forever Flashlight.

Light—when you need it the most. What makes the Forever Flashlight so unique? Perhaps it's the fact that it never needs any batteries, or new light bulbs. The Forever Flashlight uses the Faraday Principle of Electromagnetic Energy to eliminate the need for replacement bulbs and batteries. The Faraday Principle states that if an electric conductor, like a copper wire, is moved through a magnetic field, electric current will flow in the conductor. Essentially, moving the wire through the magnetic field causes electric current to flow in the wire, which is exactly how the Forever Flashlight operates. By shaking the Flashlight vigorously for 15-30 seconds, enough electricity is produced to turn the light bulb on for up to five minutes of continuous light. The super bright blue LED light is highly visible, allowing you to see exactly what you need to see, or to get someone's attention. Dimensions are 2" x 11" long.

Don't get caught in the dark. Why worry about "live" batteries in your conventional flashlight during a power outage. We're so sure you'll find the Forever Flashlight convenient and reliable, we're backing it with TechnoScout's exclusive home trial. Try this product for 30 days and return it for the full purchase price, less shipping and handling, if not completely satisfied.

Forever Flashlight

ZR-2815 ....$39.95 + S&H

Please mention promotional code **23829.**

For fastest service, call toll-free 24 hours a day

800-399-7858

To order by mail with check or money order, or by credit card, please call for total amount plus S&H. To charge it to your credit card, enclose your account number and expiration date. Virginia residents only—please add 4.5% sales tax.

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denounced it for its "denial of the reality of politics," by which he meant that it reduced widespread postwar problems to matters of individual psychology that could be solved by the application of good old-fashioned American virtues (hard work, patience, cheerfulness, and the like). Though Warshow was practically the only naysayer, his point of view came to prevail in informed circles; by 1957, when Manny Farber, writing in *Commentary*, dismissed *The Best Years of Our Lives* as "a horse-drawn truckload of liberal schmaltz," he was telling his readers something they felt they already knew. (Then as now, the only ones more disdainful of liberal ideals than conservatives were those on the extreme left.)

Skepticism is the critic's stock in trade, but I think we've become so used to feeling manipulated by movies that we instinctively distrust one that stirs something real in us. Seeing *The Best Years of Our Lives* again last year, in the aftermath of the media blitz that surrounded the September 11 observances (and amid vague new terrorism alerts and the prospect of war with Iraq), filled me with regret that today's popular culture responds to our current predicament only in ways that seem crass—witness the many commemorative books for which 9/11 represented both a marketing opportunity and a sell-by date—or, worse, ineffectual. Popular music was the first responder, and the most anticipated and publicized album was Bruce Springsteen's *The Rising*. Talk about a truckload of liberal schmaltz.

There have been plenty of new war movies in the past eighteen months, but *Black Hawk Down*, *We Were Soldiers*, and the rest were in production long before September 11, inspired by *Saving Private Ryan* and the anticipated success of *Pearl Harbor*—an epic that took in hundreds of millions of dollars at the box office without causing much of a stir culturally. *Pearl Harbor* was like a 1940s double feature: it combined action and dewy romance in a transparent attempt to please young moviegoers of both sexes and become a favorite date movie—*World War 90210*. It may also have been the first blockbuster ever to incorporate its own sequel, ending not with the bombing of Pearl Harbor, which would

CH. 31 – MAXWELL'S EQUATIONS & ELECTROMAGNETIC WAVES

MAXWELL'S EQUATIONS (1860'S) UNIFY ALL THE KNOWN FACTS ABOUT ELECTRIC & MAGNETIC FIELDS.

THEY CAN BE WRITTEN IN TWO FORMS:

1. INTEGRAL FORM

① $\int \vec{E}\cdot d\vec{A} = \dfrac{q_{enc}}{\varepsilon_0}$ (GAUSS LAW FOR $\vec{E}$)

② $\int \vec{B}\cdot d\vec{A} = 0$ (GAUSS LAW FOR $\vec{B}$; NO MAGNETIC CHARGES)

③ $\oint \vec{E}\cdot d\vec{s} = -\dfrac{d\Phi_B}{dt}$ (FARADAY LAW: INDUCTION)

④ $\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc} + \underbrace{\mu_0\varepsilon_0 \dfrac{d\Phi_B}{dt}}_{\text{DISPLACEMENT CURRENT}}$ (MAXWELL – AMPERE LAW)

2. DIFFERENTIAL FORM

① $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ (ρ = CHARGE DENSITY)

② $\vec{\nabla}\cdot\vec{B} = 0$

③ $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$

④ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$ ($\vec{J}$ = CURRENT DENSITY)

IN MODERN DIFFERENTIAL GEOMETRY NOTATION, THESE CAN BE WRITEN AS JUST 2 EQUATIONS:

$dF = 0$ (F = ELECTROMAGNETIC FIELD TENSOR)

$d{\star}F = J$

NOTES

1. ① & ③ WE HAVE SEEN BEFORE.

④ IS JUST AMPÈRE'S LAW WITH A CORRECTION TERM ADDED BY MAXWELL TO ACCOUNT FOR THE POLARIZATION IN DIELECTRICS.

2. ② IS THE MAGNETIC ANALOG OF GAUSS' LAW. WE CAN WRITE IT AS:

$$\int \vec{B} \cdot d\vec{A} = \mu_0 q_M$$

WHERE q_M = MAGNETIC CHARGE. BUT MAGNETIC CHARGES HAVE NEVER BEEN SEEN IN NATURE.

3. SET $\rho = J = 0$. THEN COMBINING ② WITH ④ AND ① WITH ③ LEADS TO A PAIR OF WAVE EQUATIONS:

$$\boxed{\begin{aligned} &\left(\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\right) \vec{E} = 0 \\ &\left(\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\right) \vec{B} = 0 \end{aligned}}$$

$$\left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

THESE DESCRIBE ELECTROMAGNETIC WAVES MOVING AT THE SPEED OF LIGHT:

$$\boxed{c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \frac{m}{s}}$$

MAXWELL CONCLUDED THAT: $\boxed{\text{LIGHT IS AN ELECTROMAGNETIC WAVE}}$

(GENERAL FORM OF A GENERIC WAVE EQUATION:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) f(x, y, z, t) = 0, \quad v = \text{SPEED OF WAVE}$$
)

FROM MAXWELL'S EQS, CAN DERIVE WAVE EQUATIONS. THESE EQUATIONS ARE SOLVED BY E & B FIELDS OF FORM:

$$\boxed{\begin{aligned} E &= E_m \sin(kx-\omega t) \\ B &= B_m \sin(kx-\omega t) \end{aligned}}$$

$$\begin{aligned} &kx-\omega t \\ &= \frac{2\pi}{\lambda}x - 2\pi f t \\ &= 2\pi\left(\frac{x}{\lambda} - ft\right) \end{aligned}$$

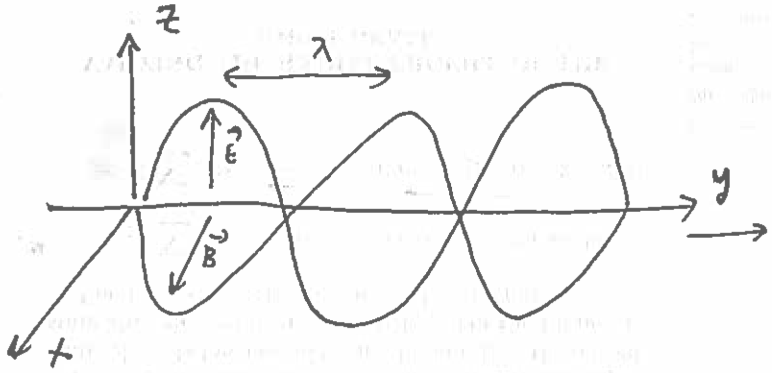
HERE, E_m & B_m = AMPLITUDES (MAXIMUM VALUES)

$\omega = 2\pi f$ = ANGULAR FREQ. (IN RAD/S)

$k = \frac{2\pi}{\lambda}$ = WAVENUMBER (IN m^{-1})

λ = WAVELENGTH (IN m)

DEFINE $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \frac{m}{s}$ = SPEED OF LIGHT (IN VACUUM)



$\vec{v}$ = VELOCITY OF WAVE ($|\vec{v}| = c$)

$\vec{E}$ & $\vec{B}$ OSCILLATE IN PHASE.

$\vec{E} \perp \vec{B} \perp \vec{v}$

NOTE! DIRECTION OF $\vec{E}$ = POLARIZATION DIRECTION

BASIC RELATIONS

$$\boxed{\frac{E_m}{B_m} = c, \qquad c = \lambda f}$$

EXAMPLES OF ELECTROMAGNETIC WAVES:

X - RAYS

ULTRAVIOLET

VISIBLE
- VIOLET
- ⋮
- RED

INFRARED

MICROWAVE

RADIO WAVES

↑ INCREASING FREQUENCY & ENERGY

↓ LONGER WAVELENGTH

VISIBLE LIGHT:

$$7.5 \times 10^{14}\,\mathrm{Hz} < f < 4.3 \times 10^{14}\,\mathrm{Hz}$$

$$400\,\mathrm{nm} < \lambda < 700\,\mathrm{nm},$$

CONVENIENT UNITS:

$$1\,\mathrm{nm} = 10^{-9}\,\mathrm{m}$$

$$1\,\mathrm{\AA} = 10^{-10}\,\mathrm{m}$$

EXAMPLES

1. A BEAM OF LIGHT (BLUE LIGHT, $\lambda = 450\ nm$) IS TRAVELING ALONG X-AXIS & IS POLARIZED IN Z-DIRECTION. ELECTRIC FIELD HAS AMPLITUDE $E_m = 10\ \frac{V}{m}$.

   a) WHAT DIRECTION IS $\vec{B}$?

   - ALONG y-AXIS.

   b) WHAT IS FREQUENCY?

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8\ \frac{m}{s}}{450 \times 10^{-9}\ m} = \boxed{6.67 \times 10^{14}\ Hz}$$

   c) WHAT IS THE MAGNETIC FIELD AMPLITUDE?

$$B_m = \frac{E_m}{c} = \frac{10\ V/m}{3 \times 10^8\ m/s} = \boxed{3.3 \times 10^{-8}\ T}$$

2. RADIO WAVES HAVE FREQUENCIES IN RANGE 3000 → 3×10^{11} Hz. WHAT IS RANGE OF WAVELENGTHS?

$$f = 3000\ Hz \rightarrow \lambda = \frac{c}{f} = \frac{3 \times 10^8}{3000} = 10^5\ m$$

$$f = 3 \times 10^{11}\ Hz \rightarrow \lambda = \frac{c}{f} = \frac{3 \times 10^8}{3 \times 10^{11}} = 10^{-3}\ m$$

$$\boxed{1\ mm < \lambda < 100\ km}$$

ENERGY DENSITY:

IN ELECTRIC FIELD: $u_E(t) = \frac{1}{2}\varepsilon_0 E^2(t)$

IN MAGNETIC FIELD: $u_B(t) = \frac{1}{2\mu_0} B^2(t)$.

TOTAL: $u(t) = \frac{1}{2}\varepsilon_0 E^2(t) + \frac{1}{2\mu_0} B^2(t)$.

BUT NOTE:

$$u_B = \frac{1}{2\mu_0} B^2 = \frac{1}{2}\left(\varepsilon_0 c^2\right)\left(\frac{E}{c}\right)^2 = \frac{1}{2}\varepsilon_0 E^2 = u_E \rightarrow \boxed{u_B = u_E}$$

SO CAN WRITE: $\boxed{u(t) = 2u_E(t) = \varepsilon_0 E^2(t)}$

RECALL ROOT-MEAN-SQUARE: $E_{RMS} = \sqrt{\overline{E^2}}$, $B_{RMS} = \sqrt{\overline{B^2}}$,

FOR SIN FUNCTIONS, $RMS = \frac{1}{\sqrt{2}}(\text{MAXIMUM})$:

$$\boxed{E_{RMS} = \frac{1}{\sqrt{2}} E_M, \quad B_{RMS} = \frac{1}{\sqrt{2}} B_M}$$

SO AVERAGE ENERGY DENSITY IS:

$$\boxed{\bar{u} = \varepsilon_0 E_{RMS}^2 = \frac{1}{2}\varepsilon_0 E_M^2} \quad \left(\text{OR } \bar{u} = \frac{1}{\mu_0} B_{RMS}^2 = \frac{1}{2\mu_0} B_M^2\right)$$

EXAMPLE:

a) TYPICAL E_M OF SUNLIGHT AT EARTH'S SURFACE: $\overline{E} = 500\ V/m$

FIND E_{RMS} & $\bar{u}$. $\rightarrow E_{RMS} = \frac{1}{\sqrt{2}}\overline{E} = \boxed{353.6\ \frac{V}{m}}$

$$\bar{u} = \frac{1}{2}\varepsilon_0 E_m^2 = \frac{1}{2}(8.85\times10^{-12})(500)^2$$

$$= \boxed{1.1\times10^{-6}\ J/m^3}$$

b) TYPICAL E_M IN LASER: $E_M = 2\times10^8\ V/m$

FIND $\bar{u}$. $\bar{u} = \frac{1}{2}\varepsilon_0 E_m^2 = \frac{1}{2}(8.85\times10^{-12})(2\times10^8)^2$

$$= \boxed{1.77\times10^5\ J/m^3}$$

COMPARE

TWO MORE DEFINITIONS:

1. **INTENSITY**: $I = \frac{\bar{P}}{A}$ = AVERAGE POWER PER AREA (IN W/m²)

3. **POYNTING VECTOR**:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

(IN W/m²)

DIRECTION: $\vec{S}$ IS ⊥ TO BOTH $\vec{E}$ & $\vec{B}$,
IN THE DIRECTION OF WAVE'S

MAGNITUDE OF $\vec{S}$:

$$S(t) = \frac{1}{\mu_0}EB = \frac{1}{c\mu_0}E^2(t)$$

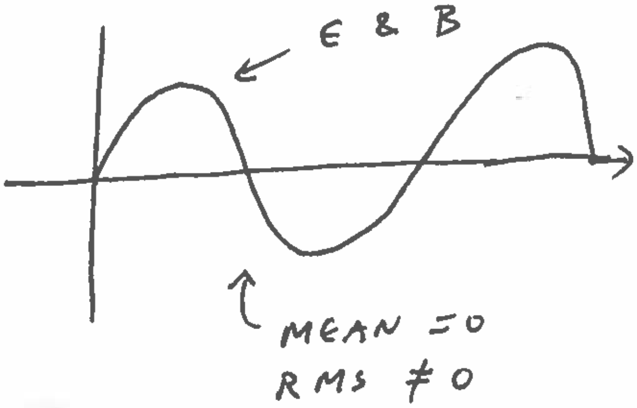
(← TIME-DEPENDENT)

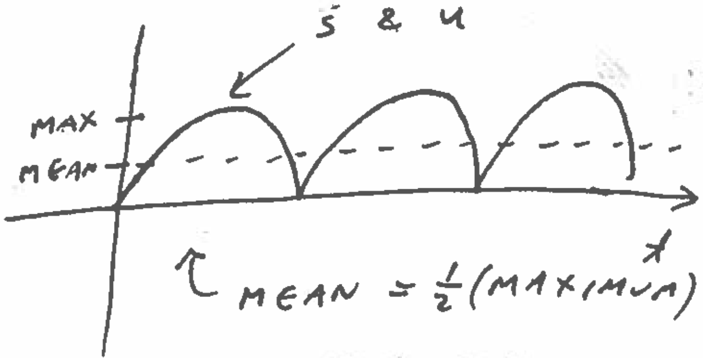
AVERAGE MAGNITUDE:

$$\bar{S} = \frac{1}{c\mu_0}E^2_{RMS} = \frac{1}{2c\mu_0}E_m^2 = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,E_m^2$$

NOTE THAT: $\bar{S} = \frac{1}{2}S_{MAX}$ IS JUST THE **INTENSITY**:

$$\boxed{\bar{S} = I = c\bar{u}}$$





EXAMPLE

$\vec{E}$ ALONG X-AXIS WITH MAGNITUDE $E_m = 100 \frac{V}{m}$.

a) MAGNITUDE OF $\vec{B}$? $B_m = \frac{E_m}{c} = \boxed{3{,}33 \times 10^{-7}\ T}$

b) IF $\vec{B}$ ALONG Y AXIS, FIND MAXIMUM VALUE OF $\vec{S}$.

$$\vec{S}_m = \frac{1}{\mu_0} \vec{E}_m \times \vec{B}_m \Longrightarrow \begin{cases} S_m = \frac{E_m B_m}{\mu_0} = \frac{(100)(3{,}33 \times 10^{-7})}{4\pi \times 10^{-7}} \\ \quad = \boxed{26{,}5 \frac{W}{m^2}} \\ \text{DIRECTION: } Z\text{-AXIS} \end{cases}$$

c) AVE. MAGNITUDE OF $\vec{S}$? $\bar{S} = S_m = \boxed{13{,}25 \frac{W}{m^2}}$

d) INTENSITY? $I = \bar{S} = \boxed{13{,}25 \frac{W}{m^2}}$

e) AVE. ENERGY DENSITY?

ONE WAY: $\bar{u} = \frac{I}{c} = \frac{13{,}25 \frac{W}{m^2}}{3 \times 10^8 \frac{m}{s}} = \boxed{4{,}4 \times 10^{-8} \frac{J}{m^3}}$

OTHER WAYS:

$$\bar{u} = \frac{1}{2} \varepsilon_0 E_m^2$$

$$\bar{u} = \frac{1}{2\mu_0} B_m^2$$

$$\bar{u} = \frac{\bar{S}}{c} = \frac{S_m}{2c}$$

POINT SOURCES: $\boxed{I = \frac{P_s}{4\pi r^2}}$ (P_s = POWER RADIATED AT SOURCE)

EXAMPLES

1. AVERAGE INTENSITY OF SUNLIGHT AT EARTH'S SURFACE:

$I = 1400 \, W/m^2$ (SOLAR CONSTANT)

a) AT WHAT RATE IS SUN RADIATING ENERGY?

$$P_s = 4\pi r^2 I = 4\pi(1.5 \times 10^{11} \, m) = \boxed{2.82 \times 10^{23} \, W}$$

$r = 1.5 \times 10^{11} \, m = 1 \, A.U.$

= AVE. EARTH-SUN DISTANCE

b) WHAT IS AVERAGE INTENSITY OF SUNLIGHT ON PLUTO?

($r = 5.9 \times 10^{12} \, m = 39.5 \, A.U.$)

$$I = \frac{P_s}{4\pi r^2} = \frac{2.82 \times 10^{23}}{4\pi(1.5 \times 10^{11})^2} = .997 \, W \, \boxed{\approx 1 \, W}$$

(1400 TIMES LESS THAN EARTH)

2. A RADIO TOWER SENDS OUT RADIO WAVES, TRANSMITTING WITH POWER OF $P_s = 50000 \, W$.

WHAT IS INTENSITY OF SIGNAL 10 km AWAY?

$$I = \frac{P_s}{4\pi r^2} = \frac{5 \times 10^4 \, W}{4\pi(10^4 \, m)^2} = \boxed{4.0 \times 10^{-5} \frac{W}{m^2}}$$

(AT 1 km $\rightarrow$ $4 \times 10^{-3} \frac{W}{m^2}$ = 4 mW/m²)

NOTE: RELATION BETWEEN I & P_s DIFFERENT FOR NON-POINT SOURCES.

EXAMPLE: $P_s = 1 \, mW$ LASER, WITH BEAM OF RADIUS $r = .5 \, mm$.

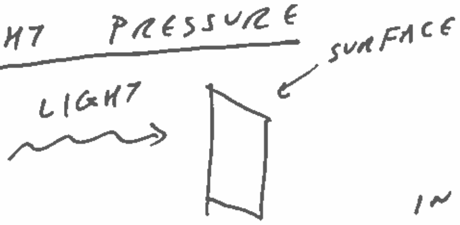
$$\rightarrow a) \; I = \frac{P_s}{A} = \frac{10^{-3} \, W}{\pi(5 \times 10^{-4} \, m)^2} = \boxed{1.27 \times 10^3 \frac{W}{m^2}}$$

≈ INDEPENDENT OF DISTANCE.

b) ENERGY IN 10 s?

$$\text{ENERGY} = P_s \Delta t = (10^{-3} \, W)(10 \, s) = \boxed{.01 \, J}$$

LIGHT PRESSURE



LIGHT CARRIES MOMENTUM:

$$p = \frac{IA\Delta t}{c} \quad \text{(IN TIME } \Delta t\text{)}$$

WHEN LIGHT STRIKES SURFACE, CHANGE IN MOMENTUM LEADS TO FORCE:

$$F = \frac{\Delta p}{\Delta t} = \begin{cases} \frac{IA}{c}, & \text{ABSORPTION} \\ \frac{2IA}{c}, & \text{REFLECTION} \end{cases}$$

SO LIGHT EXERTS PRESSURE:

RADIATION PRESSURE:
$$\boxed{P_r = \begin{cases} \frac{I}{c} & \text{(ABSORPTION)} \\ \frac{2I}{c} & \text{(REFLECTION)} \end{cases}}$$

EXAMPLE:

SUNLIGHT FALLS ON SOLAR SAIL AT DISTANCE $r = 10^{11}\,m$ FROM SUN. FIND AREA NEEDED TO ACCELERATE $M = 10^5\,kg$ SHIP AT $1\,m/s^2$, ASSUMING SAIL IS a) BLACK b) WHITE.

$$P_s = 2.8 \times 10^{23}\,W \longrightarrow I = \frac{P_s}{4\pi r^2} = \frac{2.8 \times 10^{23}}{4\pi (10^{11})^2} = 2.23\,\frac{W}{m^2}$$

SO:

a) BLACK (ABSORBING): $P_r = \frac{I}{c} = 7.43 \times 10^{-9}\,\frac{N}{m^2}$.

NECESSARY FORCE: $F = ma = (10^5)(1) = 10^5\,N$

SO: $$A = \frac{F}{P_r} = \frac{10^5}{7.43 \times 10^{-9}} = \boxed{1.35 \times 10^{13}\,m^2}$$

b) WHITE (REFLECTING): NOW PRESSURE IS TWICE AS LARGE: $P_r = \frac{2I}{c} = 1.48 \times 10^{-8}\,\frac{N}{m^2}$.

SO AREA IS HALF AS LARGE:

$$A = \frac{F}{P_r} = \boxed{6.75 \times 10^{12}\,m^2}$$

STILL BIG!

- IN 2019, A 32M LIGHT SAIL WAS DEPLOYED BY THE PLANETARY SOCIETY. IT WAS PLANNED TO KEEP SATELITE IN ORBIT FOR 1 YEAR, MISSION HAS NOW BEEN EXTENDED TO 2 YEARS.
- THE FIRST MENTION OF PROPELLING A SHIP BY USING LIGHT WAS APPARENTLY BY JULES VERNE (FROM THE EARTH TO THE MOON)
- RADIATION PRESSURE PLAYS A LARGE ROLE IN STAR FORMATION AND NEBULA FORMATION.
- IT HAS BECOME AN IMPORTANT TOOL IN NANOTECHNOLOGY; RADIATION PRESSURE FROM A LASER ALLOWS THE MANIPULATION OF INDIVIDUAL MOLECULES TO FORM NANOSCALE STRUCTURES VIA OPTICAL TWEEZERS.

CH. 32 – REFLECTION AND REFRACTION

LIGHT IS AN ELECTROMAGNETIC WAVE (SEE CH. 22), BUT UNDER MOST EVERYDAY CIRCUMSTANCES WE CAN TREAT IT AS A RAY OF PARTICLES TRAVELLING IN STRAIGHT LINES. THIS PICTURE IS CALLED GEOMETRICAL OPTICS. (SOME WAVE ASPECTS OF LIGHT WILL BE EXAMINED IN THE NEXT CHAPTER)

SPEED OF LIGHT IN VACUUM: $c = 3 \times 10^8 \frac{m}{s}$ (APPROX.)

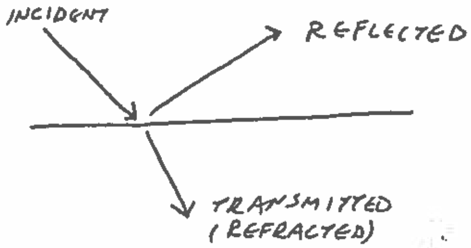
INSIDE MATTER, LIGHT TRAVELS MORE SLOWLY, AT SPEED v.

INDEX OF REFRACTION: $n = \frac{c}{v}$, DEPENDS ON MATERIAL.

NOTE THAT $n \geq 1$. $n_{VACUUM} = 1 \approx n_{AIR}$

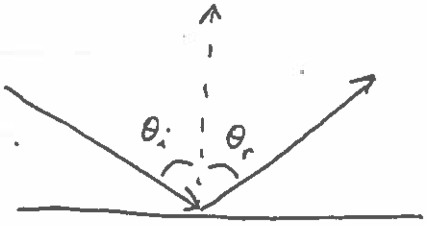
VALUES OF n ARE GIVEN IN TABLE 23-1, p. 646.

SUPPOSE LIGHT TRAVELLING THROUGH MATERIAL OF ONE INDEX STRIKES MATERIAL OF A DIFFERENT INDEX THEN PART OF THE LIGHT IS TRANSMITTED TO THE NEW MATERIAL & PART IS REFLECTED.



I. LETS LOOK AT THE REFLECTED LIGHT FIRST.

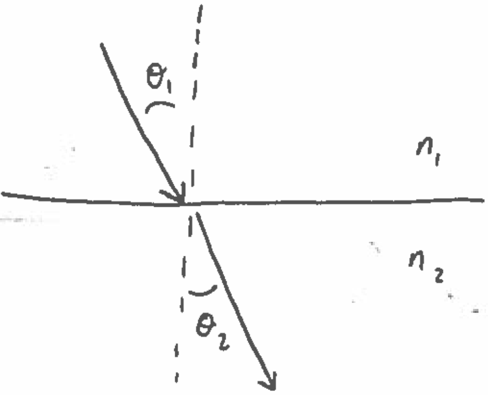
θ_i, θ_r = ANGLES OF INCIDENCE & OF REFLECTION RELATIVE TO LINE ⊥ TO SURFACE.



LAW OF REFLECTION: $\theta_i = \theta_r$

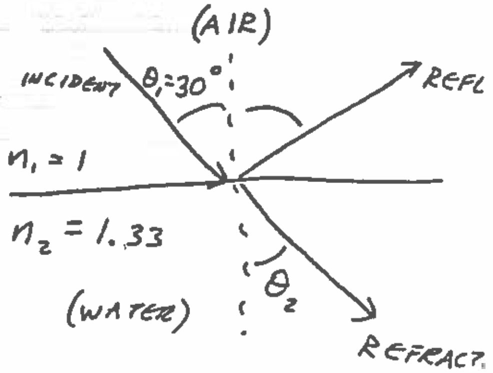
II. NOW FOR REFRACTED LIGHT.

SNELL'S LAW: $n_1 \sin\theta_1 = n_2 \sin\theta_2$



EXAMPLES

1. A BEAM OF LIGHT STRIKES THE SURFACE OF A POND AT 30° FROM VERTICAL. THE INDEX OF REFRACTION OF WATER IS 1.33.



a) WHAT ANGLE IS THE REFLECTED BEAM?

$$\theta_r = \theta_i = \boxed{30°}$$

b) AT WHAT ANGLE IS THE REFRACTED BEAM?

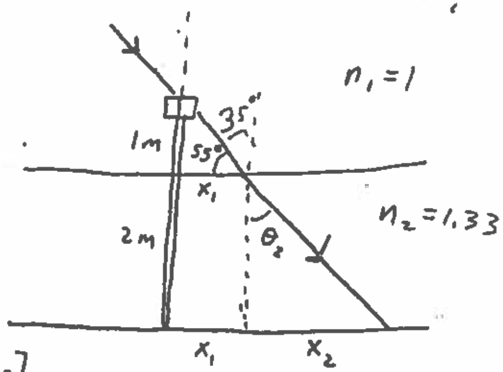
$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \Longrightarrow \theta_2 = \sin^{-1}\left[\frac{n_1}{n_2}\sin\theta_1\right]$$

$$= \sin^{-1}\left[\left(\frac{1}{1.33}\right)\sin 30°\right] = \boxed{22.08°}$$

c) WHAT IS THE SPEED OF LIGHT IN WATER?

$$V = \frac{c}{n} = \frac{3\times10^8 \frac{m}{s}}{1.33} = \boxed{2.26\times10^8 \frac{m}{s}}$$

-. A "NO SWIMMING" SIGN EXTENDS 1 m ABOVE SURFACE OF A POOL 2 m DEEP. THE SUN IS 55° ABOVE THE HORIZON. WHAT IS THE LENGTH OF THE SHADOW ON THE POOL'S BOTTOM?



DEFINE X_1 & X_2 AS IN DIAGRAM.

$$X_1 = \frac{1m}{\tan 55°} = \boxed{.70 m}$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \leftarrow \text{NOTE: } \theta_1 = 35°, \text{ NOT } 55°.$$

$$\theta_2 = \sin^{-1}\left[\frac{n_1}{n_2}\sin\theta_1\right] = \sin^{-1}\left[\frac{1}{1.33}\sin 35°\right] = 25.55°.$$

$$X_2 = \frac{\tan\theta_2}{2m} = \frac{\tan 25.55°}{2m} = .239 m$$

$$\therefore \quad L = X_1 + X_2 = .70 + .239 = \boxed{.939 m}$$

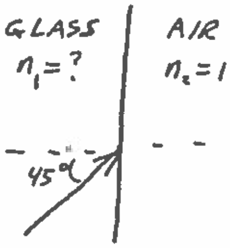
TOTAL INTERNAL REFLECTION: SUPPOSE $n_2 < n_1$

THEN IT IS POSSIBLE FOR BEAM TO ALL BE REFLECTED, WITH NONE TRANSMITTED. THIS OCCURS WHEN $\theta_2 = 90°$. THE MINIMUM INCIDENT ANGLE AT WHICH THIS OCCURS IS THE CRITICAL ANGLE, θ_c. FIND θ_c USING SNELL'S LAW;

$$n_1 \sin\theta_c = n_2 \underbrace{\sin 90°}_{1} \implies \boxed{\theta_c = \sin^{-1}\frac{n_2}{n_1}}$$

THIS IS THE BASIS OF FIBRE OPTICS. IF θ_c IS LARGE, THEN θ_1 WILL RARELY EXCEED θ_c & SO NO LIGHT IS LOS

EXAMPLE: A BEAM OF LIGHT INSIDE A GLASS PRISM HITS THE SURFACE AT $\theta_1 = 45°$.



a) IF AIR IS OUTSIDE THE PRISM, WHAT IS MINIMUM INDEX OF REFRACTION, n THAT WILL LEAD TO TOTAL INTERNAL REFLECTION?

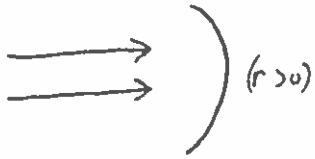
$$\theta > \theta_c \implies \sin\theta > \sin\theta_c = \frac{n_2}{n_1}$$

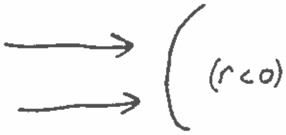
$$\text{SO, } n_1 > \frac{n_2}{\sin\theta} = \frac{1}{\sin 45} = \boxed{1.414}$$

b) WHAT IS MINIMUM n_1 IF THE PRISM IS PLACED IN WATER?

$$n_1 > \frac{n_2}{\sin\theta} = \frac{1.33}{\sin 45°} = \boxed{1.881}$$

SPHERICAL MIRRORS





r = RADIUS OF CURVATURE

i) $r > 0 \Longrightarrow$ CONCAVE MIRROR

ii) $r < 0 \Longrightarrow$ CONVEX MIRROR

iii) $r = \infty \Longrightarrow$ PLANE MIRROR

LIGHT RAYS PARALLEL TO THE AXIS ALL FOCUS AT A SINGLE POINT. THE DISTANCE FROM THIS FOCAL POINT TO THE MIRROR IS THE FOCAL LENGTH, f.

RELATION BETWEEN r & f: $\boxed{f = \frac{r}{2}}$ FOR MIRRORS ONLY NOT FOR LENSES.

NOW PLACE AN OBJECT IN FRONT OF THE MIRROR, & SEE WHERE ITS IMAGE APPEARS TO BE.

d_o = OBJECT DISTANCE (POSITIVE)

d_i = IMAGE DISTANCE

$d_i > 0 \Longrightarrow$ REAL IMAGE: LIGHT PASSES THROUGH IMAGE POINT

$d_i < 0 \Longrightarrow$ VIRTUAL IMAGE: LIGHT DOES NOT REACH IMAGE POIN

m = MAGNIFICATION

$|m| > 1 \longrightarrow$ IMAGE IS LARGER THAN OBJECT

$|m| < 1 \longrightarrow$ IMAGE IS SMALLER

$m < 0 \longrightarrow$ IMAGE IS UPSIDE-DOWN

MAIN EQUATIONS:

1. $$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$$

2. $$m = \frac{h_i}{h_o} = -\frac{d_i}{d_o}$$

(h_o, h_i = HEIGHTS OF OBJECT & IMAGE)

NOTE: FOR A PLANE MIRROR,

$R = \infty$ $\quad d_i = -d_o$ $\quad$ (ALL IMAGES VIRTUAL)

$f = \infty$ $\quad M = 1$

EXAMPLES

1. A CONCAVE MIRROR HAS RADIUS OF CURVATURE $r = .22\,m$.

a) IF AN OBJECT IS PLACED .14 m IN FRONT OF MIRROR, WHERE IS ITS IMAGE?

$$\left.\begin{aligned} f &= \frac{r}{2} = +.11\,m \\ d_o &= .14\,m \end{aligned}\right\} \longrightarrow \frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \rightarrow d_i = \frac{1}{\frac{1}{f} - \frac{1}{d_o}}$$

$$= \frac{1}{\left(\frac{1}{+.11}\right) - \left(\frac{1}{.14}\right)} = \boxed{+.513\,m}$$

(IN FRONT OF MIRROR)

b) IS THE IMAGE REAL OR VIRTUAL?

$$d_i > 0 \Longrightarrow \boxed{\text{REAL}}$$

c) WHAT IS THE MAGNIFICATION?

$$m = -\frac{d_i}{d_o} = -\frac{.513}{.14} = \boxed{-3.66} \Rightarrow$$ IMAGE IS LARGER & UPSIDE-DOWN

d) IF THE OBJECT BEING VIEWED IS 2 cm TALL, HOW TALL IS ITS IMAGE?

$$h_i = m h_o = (-3.66)(2\,cm) = \boxed{-7.32\,cm}$$

UPSIDE-DOWN

2. REPEAT PREVIOUS EXAMPLE FOR A CONVEX MIRROR.

a) NOW, $r = -.22\,m \Longrightarrow f = \frac{r}{2} = -.11\,m$.

SO: $$d_i = \frac{1}{\frac{1}{f} - \frac{1}{d_o}} = \frac{1}{\left(\frac{1}{-.11}\right) - \left(\frac{1}{.14}\right)} = \boxed{-.062\,m}$$

(BEHIND MIRROR)

b) $$d_i < 0 \longrightarrow \boxed{\text{VIRTUAL}}$$

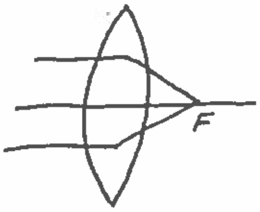
c) $$m = -\frac{d_i}{d_o} = -\left(\frac{-.062}{+.14}\right) = \boxed{+.44}$$ (UPRIGHT, SHRUNKEN IMAGE)

d) $$h_i = m h_o = (+.44)(2\,cm) = \boxed{.88\,cm}$$

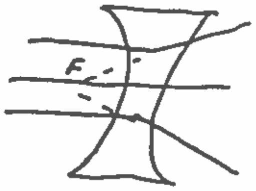
CH 33 – LENSES & OPTICAL INSTRUMENTS

LENSES

CONVERGING LENS:
$(f > 0)$



DIVERGING LENS:
$(f < 0)$



THE IMAGE WILL BE REAL ($d_i > 0$) IF IT IS ON SIDE OPPOSITE T
INCOMING LIGHT, IT WILL BE VIRTUAL ($d_i < 0$) OTHERWISE.

BASIC EQUATIONS: SAME AS FOR MIRRORS –

$$1.\quad \frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \qquad 2.\quad m = \frac{h_i}{h_o} = -\frac{d_i}{d_o}$$

NOTE: THERE ARE 2 RADII OF CURVATURE, SO $f = \frac{r}{2}$ IS NOT TRUE.

POWER OF A LENS: $P = \frac{1}{f}$, IN DIOPTERS (D). $1\ D = 1\ m^{-1}$

NOTE THAT THERE ARE 2 FOCAL POINTS, ONE FOR
LIGHT COMING FROM EACH DIRECTION.

RAY DIAGRAMS CAN BE DRAWN AS FOR MIRRORS.

SIGN CONVENTIONS (MIRRORS & LENSES)

$d_o \Rightarrow$ POSITIVE WHEN OBJECT IS ON SIDE OF INCOMING LIGHT

$d_i \Rightarrow$ POSITIVE WHEN IMAGE IS ON SIDE OF OUTGOING LIGHT

$R \Rightarrow$ POSITIVE WHEN CENTER OF CURVATURE IS ON OUTGOING SI

$f \Rightarrow$ {
HAS SAME SIGN AS R (MIRROR)
POSITIVE FOR CONVERGING LENS
NEGATIVE FOR DIVERGING LENS

RAY DIAGRAMS FOR LENSES

	INCOMING RAY	OUTGOING RAY
1.	PARALLEL TO AXIS	PASSES THROUGH FOCAL PT. c OPPOSITE (SAME) SIDE FOR CONVERGING (DIVERGING) LENS
2.	COMES FROM FOCAL PT. ON SAME (OPPOSITE) SIDE AS OBJECT FOR CONV. (DIV.) LENS	GOES OUT // TO AXIS
3.	GOES TOWARD CENTER	CONTINUES IN STRAIGHT LINE

EXAMPLES

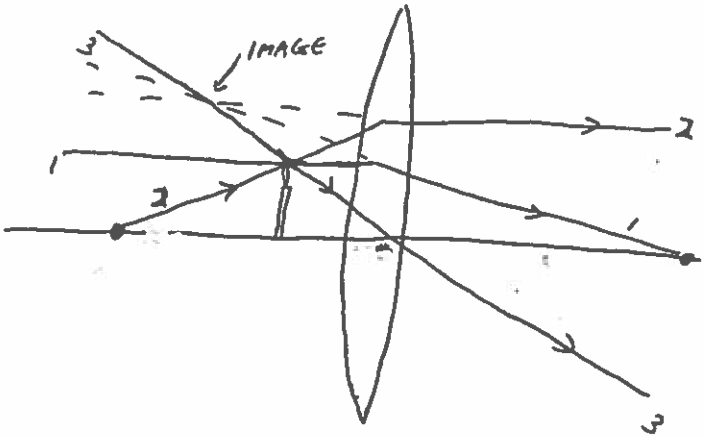
1. AN OBJECT IS AT $d_o = .095\ m$ TO LEFT OF LENS OF $P = 4.167\ D$.

a) FOCAL DISTANCE?

$$f = \frac{1}{P} = \boxed{.24\ m}$$

b) CONVERGING OR DIVERGING?

$$f > 0 \Rightarrow \boxed{\text{CONVERGING}}$$



c) FIND IMAGE.

$$d_i = \frac{1}{\frac{1}{f} - \frac{1}{d_o}} = \frac{1}{\frac{1}{.24} - \frac{1}{.095}} = \boxed{-.157\ m}$$

TO LEFT

d) REAL OR VIRTUAL?

$$d_i < 0 \Rightarrow \boxed{\text{VIRTUAL}}$$

e) MAGNIFICATION?

$$m = -\frac{d_i}{d_o} = -\frac{(-.157)}{.095} = \boxed{+1.65}$$

(UPRIGHT)

2. REPEAT FOR POWER $P = -4.167\ D$.

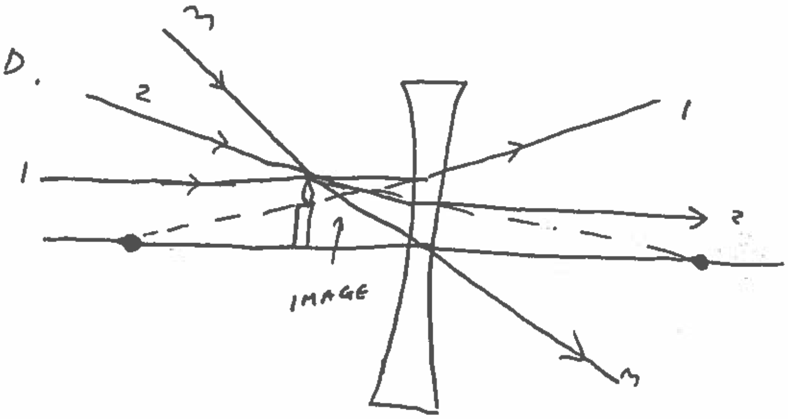
a) $f = \frac{1}{P} = \boxed{-.24\ m}$

b) $f < 0 \Rightarrow \boxed{\text{DIVERGING}}$

c) $d_i = \left(\frac{1}{\frac{1}{-.24} - \frac{1}{.095}}\right) = \boxed{-.068\ m}$

TO LEFT

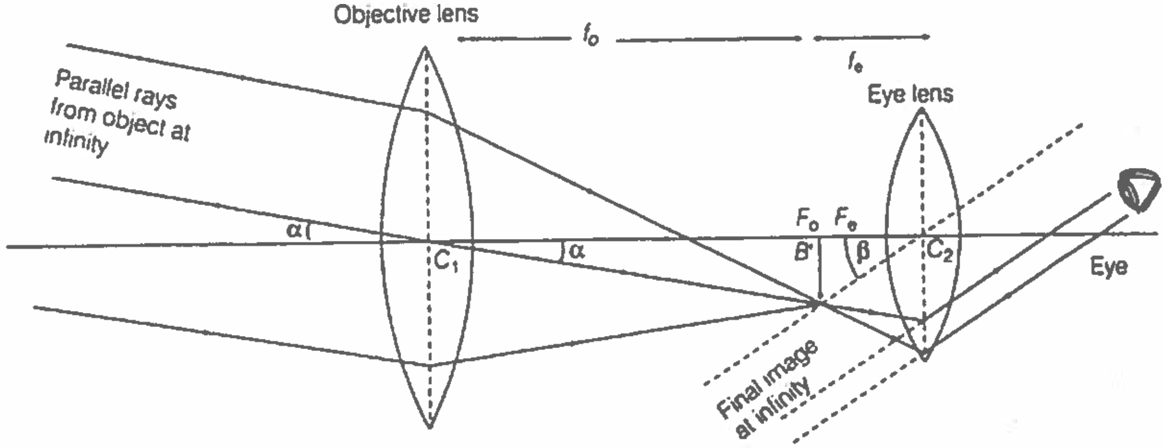
d) $\boxed{\text{VIRTUAL}}$



e) $m = -\frac{d_i}{d_o} = -\left(\frac{-.068}{.095}\right) = \boxed{.716}$

(UPRIGHT)

TELESCOPE: OBJECT FAR AWAY → INCIDENT RAYS ALL PARALLEL TO EACH OTHER


Objective lens
f_o
f_e
Eye lens
Parallel rays from object at infinity
α
C_1
α
F_o F_e
B'
β
C_2
Eye
Final image at infinity

$f_o + f_e$ = DISTANCE BETWEEN LENSES

MAGNIFICATION: $\boxed{M = -\frac{f_o}{f_e}}$

FOR HIGH MAGNIFICATION, WANT SHORT f_e, LONG f_o.

EXAMPLE: YERKES OBSERVATORY (WISCONSIN)
40 INCH APERTURE

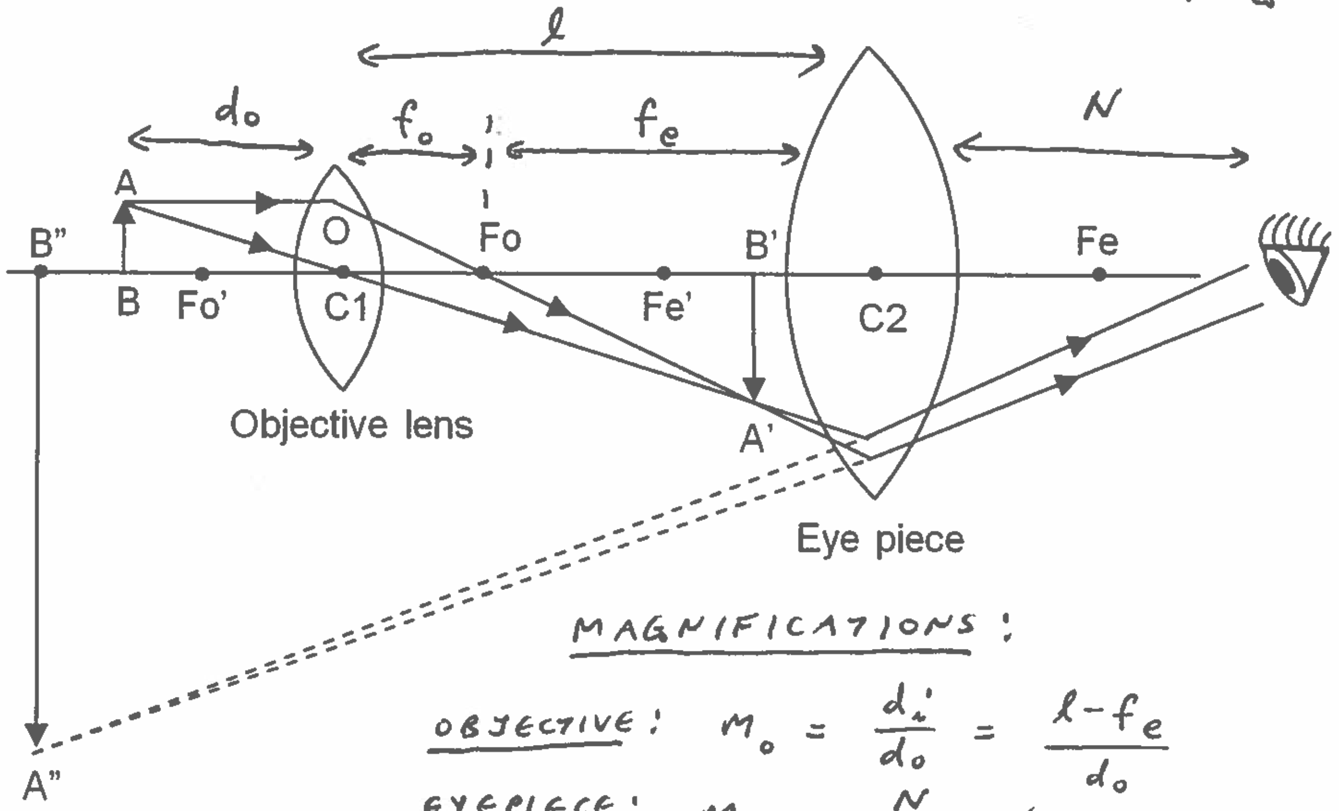
$$M = -\frac{f_o}{f_e} = \frac{-19\,m}{.10\,m} = \boxed{-190}$$

WANT LARGE APERTURES FOR 2 REASONS:

1. ALLOW MORE LIGHT TO ENTER → CAN SEE DIMMER OBJECTS.

2. LESS DISRUPTION FROM DIFFRACTION AT EDGE OF APERTURE → HIGHER RESOLUTION

MICROSCOPE: INCIDENT RAYS STRONGLY DIVERGING


l
d_o
f_o
f_e
N
A
B"
O
Fo
B'
Fe
B
Fo'
C1
Fe'
C2
Objective lens
A'
Eye piece
A"

MAGNIFICATIONS:

OBJECTIVE: $m_o = \frac{d_i}{d_o} = \frac{l - f_e}{d_o}$

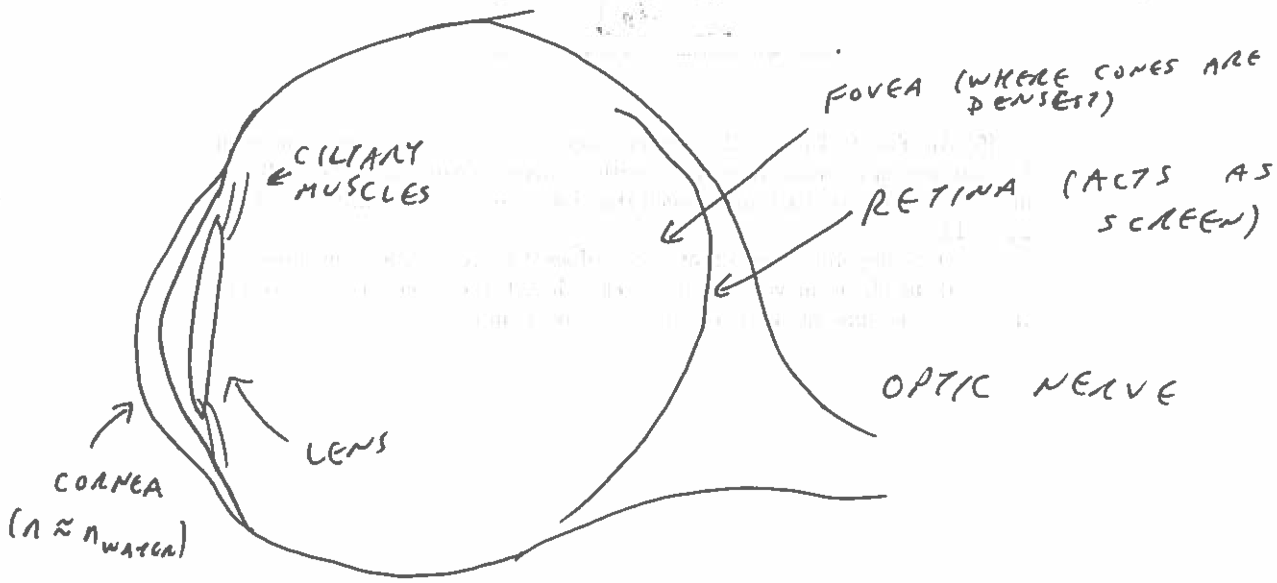
EYEPIECE: $m_e = \frac{N}{f_e}$ (N = NEAR POINT DISTANCE)

TOTAL: $M = m_o m_e = \left(\frac{l-f_e}{d_o}\right)\left(\frac{N}{f_e}\right)$

$$\downarrow$$

$$\boxed{M \approx \frac{Nl}{f_e f_o}} \qquad \left(\begin{matrix} f_e \ll l \\ f_o \approx d_o \end{matrix}\right)$$

THE EYE



FOVEA (WHERE CONES ARE DENSEST)
CILIARY MUSCLES
RETINA (ACTS AS SCREEN)
OPTIC NERVE
LENS
CORNEA
$(n \approx n_{WATER})$

IMAGES FOCUS ON RETINA. THE FOCUSING IS PROVIDED BY LENS & CORNEA ACTING TOGETHER.

$$\left.\begin{array}{ll}\text{CORNEA: } f = 2.4\text{ cm} \rightarrow P_c = 42\text{ D} \\ \text{LENS: } f = 8.3\text{ cm} \rightarrow P_L = 12\text{ D}\end{array}\right\} \rightarrow P_{TOTAL} \approx P_c + P_L \approx 54\text{ D}$$

NEAR POINT: CLOSEST DISTANCE AT WHICH OBJECTS CAN BE SEEN CLEARLY

FAR POINT: FARTHEST DISTANCE.

FOR HEALTHY EYES: $\underbrace{\text{NEAR POINT} \approx 25\text{ cm}}_{\text{LENS HIGHLY CURVED}}$, $\underbrace{\text{FAR POINT} \approx \infty}_{\text{LENS IS FLAT}}$

CILIARY MUSCLES: BEND LENS TO CHANGE FOCAL LENGTH, SO EYE CAN FOCUS AT DIFFERENT DISTANCES. (ACCOMODATION)

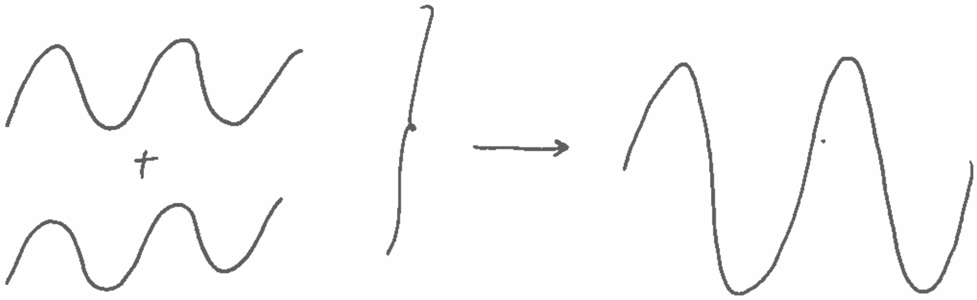
CH. 34/35: INTERFERENCE, DIFFRACTION, & POLARIZATION

RECALL, ELECTRIC FIELD IN LIGHT WAVE:

$$E(x,t) = E_m \sin(kx - \omega t) \qquad \left(k = \frac{2\pi}{\lambda}\right)$$

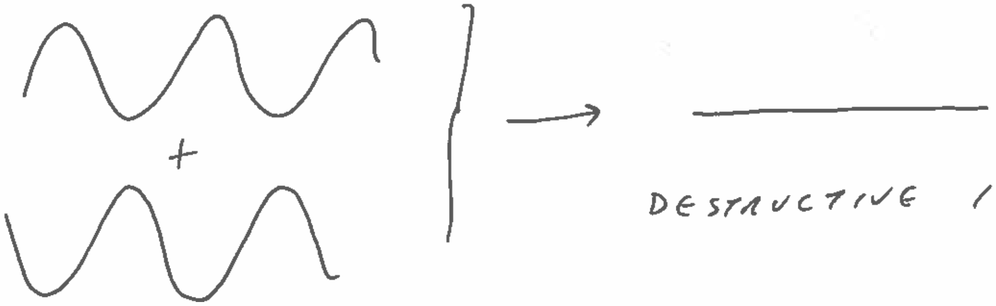
(SIMILAR EXPRESSION FOR $B(x,t)$)

SHIFTING $X \to X + \lambda$ LEAVES E UNCHANGED:



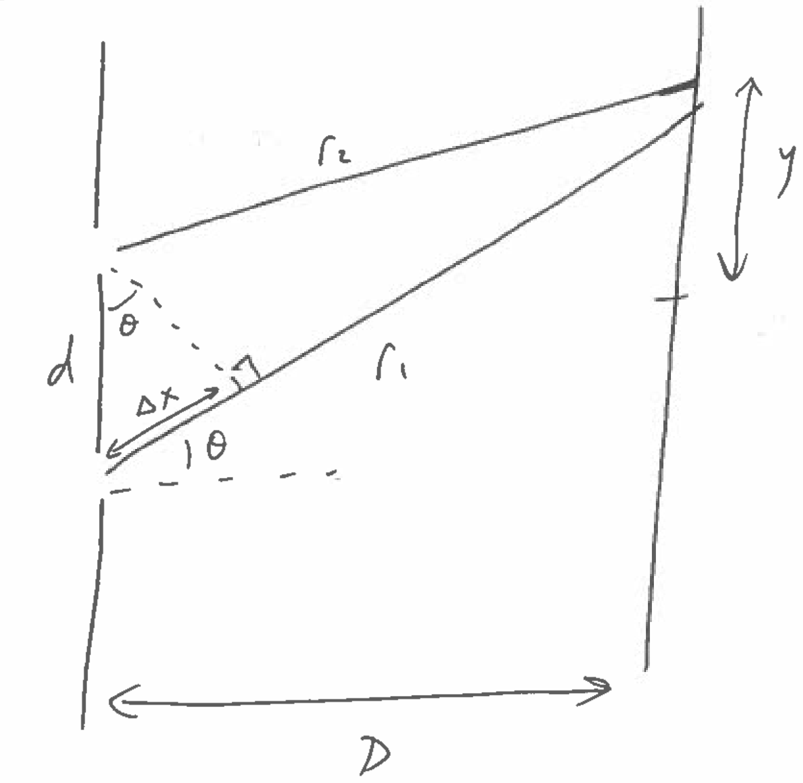
CONSTRUCTIVE INTERFERENCE

SHIFTING $X \to X + \frac{\lambda}{2}$:



DESTRUCTIVE INTERFERENCE

YOUNG TWO-SLIT EXPERIMENT



EXTRA PATH LENGTH OF LOWER BEAM:

$\Delta x = d \sin\theta.$

FOR CONSTRUCTIVE INTERFERENCE:

Δx = INTEGER # OF WAVELENGTHS

⇓

$$\boxed{d\sin\theta = m\lambda}$$

$m = 0, \pm 1, \pm 2, \ldots$

↑

BRIGHT FRINGES

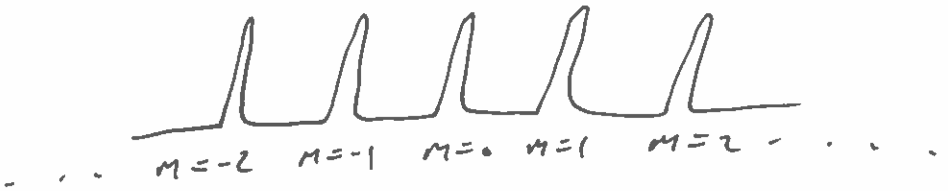
DARK FRINGES $\frac{1}{2}$ WAY BETWEEN:

$$\boxed{d\sin\theta = \left(m+\tfrac{1}{2}\right)\lambda}$$

DIFFRACTION GRATING: MANY (N) LINES OR SLITS.

MAXIMA OCCUR WHEN! $\boxed{d \sin\theta = m\lambda}$

m = ORDER = $0, \pm 1, \pm 2, \ldots$



AS N INCREASES, THE MAXIMA BECOME NARROWER.

CAN USE AS GRATING SPECTROMETER! MEASURE ANGLES, & USE FORMULA TO FIND WAVELENGTHS (& FREQUENCIES) OF THE INCOMING LIGHT.

EXAMPLE

a) LIGHT OF $\lambda = 400\text{ nm} = 4 \times 10^{-7}\text{ m}$ FALLS ON GRATING WITH ~~SPAC~~ 3000 LINES/CM. FIND ANGLES OF 1ST & 2nd ORDER MAXIMA.

$$d = \frac{1}{3000\text{ cm}^{-1}} = 3.33 \times 10^{-4}\text{ cm} = 3.33 \times 10^{-6}\text{ m}$$

$$m = 1 \rightarrow \quad \theta = \sin^{-1}\frac{m\lambda}{d} = \sin^{-1}\frac{(1)(4\times10^{-7})}{3.33\times10^{-6}} = \boxed{6.9^\circ}$$

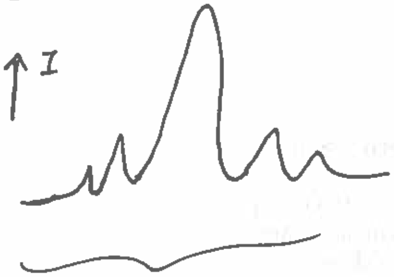
$$m = 2 \rightarrow \quad \theta = \sin^{-1}\frac{m\lambda}{d} = \sin^{-1}\frac{(2)(4\times10^{-7})}{3.33\times10^{-6}} = \boxed{13.9^\circ}$$

b) LIGHT OF UNKNOWN WAVELENGTH STRIKES SAME GRATING. FIRST ORDER MAXIMUM AT 8°. $\lambda = ?$

$$\lambda = \frac{d \sin\theta}{m} = \frac{3.33\times10^{-6} \sin 8^\circ}{1} = 4.63\times10^{-7}\text{ m}$$

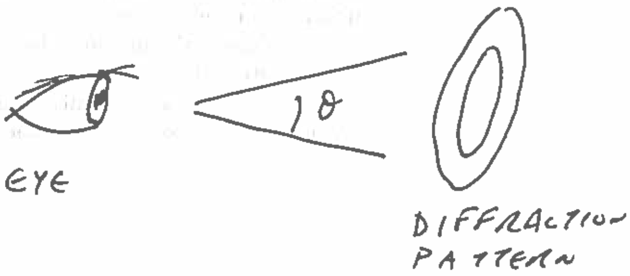
$$= \boxed{463\text{ nm}}$$

DIFFRACTION BY CIRCULAR APERTURE



BRIGHT SPOT SURROUNDED BY BRIGHT & DARK RINGS.

d = DIAMETER OF OPTICAL INSTRUMENT (MICROSCOPE, TELESCOPE, EYE, ETC.)


EYE
θ
DIFFRACTION PATTERN

FIRST MINIMUM (FIRST DARK RING):

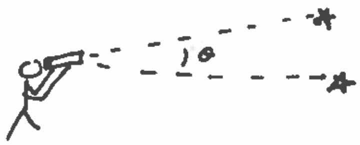
$$\boxed{\sin\theta = 1.22\,\frac{\lambda}{d}}$$

THIS GIVES A WAY OF MEASURING RESOLUTION.

RESOLUTION OR RESOLVING POWER.

THE ANGULAR RESOLUTION OF AN OPTICAL INSTRUMENT (THE EYE, A LENS, A TELESCOPE, ETC.) IS THE SMALLEST ANGULAR SEPERATION TWO OBJECTS CAN HAVE AND STILL BE DISTINGUISHED AS SEPERATE OBJECTS WHEN VIEWED THROUGH THE INSTRUMENT.

FOR EXAMPLE, SUPPOSE A TELESCOPE HAS AN ANGULAR RESOLUTION OF $.001°$. THEN TWO STARS SEPERATED BY AN ANGLE $\theta = .0005°$ WILL BLUR TOGETHER AND SEEM TO BE A SINGLE STAR.



FOR LIGHT OF WAVELENGTH λ VIEWED THROUGH AN INSTRUMENT OF APERTURE DIAMETER d, THE ANGULAR RESOLUTION IS GIVEN BY RAYLEIGH'S CRITERION:

$$\boxed{\theta = \frac{1.22\lambda}{d}}$$

(IN RADIANS)

EXAMPLE: FOR LIGHT OF $\lambda = 500\,nm$, COMPARE THE ANGULAR RESOLUTION OF THE EYE (PUPIL DIAMETER = 5 mm) TO THAT OF A SMALL TELESCOPE (APERTURE DIAMETER = 10 cm).

$$\text{(i) } \underline{\text{EYE}}: \left.\begin{array}{l} \lambda = 5\times10^{-7}m \\ d = 5\times10^{-3}m \end{array}\right\} \longrightarrow \theta = \frac{1.22\lambda}{d} = \frac{1.22(5\times10^{-7}m)}{5\times10^{-3}m} = \boxed{1.2\times10^{-4}\,RAD}$$

$$\text{(ii) } \underline{\text{TELESCOPE}}: \left.\begin{array}{l} \lambda = 5\times10^{-7}m \\ d = .1m \end{array}\right\} \rightarrow \theta = \frac{1.22\lambda}{d} = \frac{1.22(5\times10^{-7})}{.1} = \boxed{6\times10^{-6}\,RAD}$$

SO EVEN A VERY SMALL TELESCOPE HAS A RESOLUTION 20 TIMES BETTER THAN THE NAKED EYE. LARGER TELESCOPES HAVE MUCH SMALLER RESOLUTIONS.

NOTE: THE VALUE FOR THE EYE GIVEN ABOVE IS ACTUALLY OVERLY OPTIMISTIC. TWO IMAGES MUST BE SEPERATED BY AT LEAST ONE UNEXCITED ROD OR CONE TO BE RESOLVED. THE MINIMUM SPACING BETWEEN CONES IS ABOUT $2\times10^{-6}\,M$ $\Rightarrow$ ACTUAL RESOLVING POWER OF EYE IS NEVER BETTER THAN 2×10^{-4} RAD.

POLARIZATION

LIGHT CONSISTS OF OSCILLATING $\vec{E}$ & $\vec{B}$ FIELDS, $\perp$ TO EACH OTHER & TO DIRECTION OF MOTION.

MOST LIGHT IS UNPOLARIZED: DIRECTION OF $\vec{E}$ FLUCTUATES RANDOMLY.

BUT LIGHT CAN ALSO BE POLARIZED: ALL $\vec{E}$ FIELDS IN SAME DIRECTION (THE POLARIZATION DIRECTION).

LIGHT CAN BE POLARIZED IN TWO WAYS:

1. PASSING IT THROUGH MATERIAL WITH ASYMMETRIC CRYSTAL STRUCTURE (TOURMALINE, POLAROID SHEETS) WHICH BLOCK ONE POLARIZATION.
   E. LAND 1932

   (POLAROID SUNGLASSES CUT INTENSITY IN HALF BY BLOCKING ONE POLARIZATION.)

2. BY REFLECTION. REFLECTED LIGHT IS COMPLETELY POLARIZED (∥ TO SURFACE) IF INCIDENT LIGHT EQUALS THE POLARIZING ANGLE OR BREWSTER'S ANGLE:

$$\boxed{\tan \theta_p = \frac{n_2}{n_1}}$$

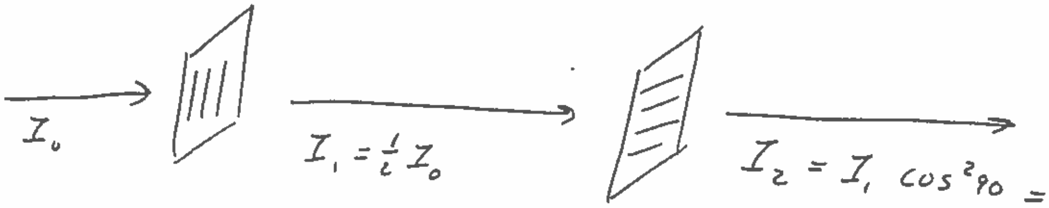
LAW OF MALUS: TWO POLARIZERS ARE AT ANGLE θ TO EACH OTHER. LIGHT LEAVES FIRST WITH INTENSITY I_1. IT LEAVES SECOND ONE WITH:

$$\boxed{I_2 = I_1 \cos^2\theta}$$

EXAMPLES

1. UNPOLARIZED LIGHT OF INITIAL INTENSITY I_0 PASSES THROUGH TWO FILTERS AT 90° TO EACH OTHER.

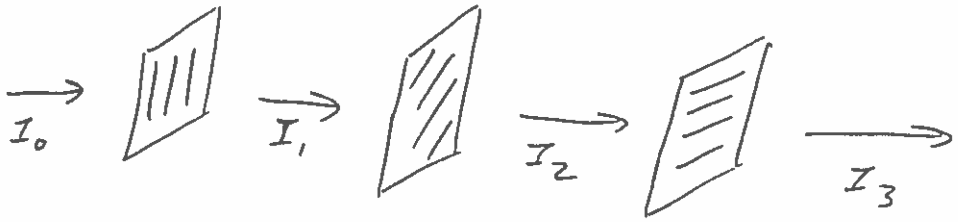
a) WHAT IS INTENSITY AFTER PASSING THROUGH EACH?



$I_1 = \frac{1}{2} I_0$

$I_2 = I_1 \cos^2 90 = 0$

b) NOW INSERT A THIRD FILTER BETWEEN THEM, AT 45°. FIND INTENSITIES.



$I_1 = \boxed{\frac{1}{2} I_0}$

$I_2 = I_1 \cos^2 45 = \left(\frac{1}{2} I_0\right)\left(\frac{1}{\sqrt{2}}\right)^2 = \boxed{\frac{1}{4} I_0}$

$I_3 = I_2 \cos^2 45 = \left(\frac{1}{4} I_0\right)\left(\frac{1}{\sqrt{2}}\right)^2 = \boxed{\frac{1}{8} I_0}$

② a) A BEAM OF LIGHT STRIKES THE SURFACE OF WATER FROM AIR. WHAT IS MIN. ANGLE FOR REFLECTED BEAM TO BE POLARIZED?

$n_1 = 1, \quad n_2 = 1.33 \implies \theta_p = \tan^{-1} \frac{n_2}{n_1} = \tan^{-1} 1.33 = \boxed{53.06°}$

b) SUPPOSE THE BEAM STARTS IN THE WATER & STRIKES THE AIR. FIND θ_p.

$n_1 = 1.33, \quad n_2 = 1 \implies \theta_p = \tan^{-1} \frac{1}{1.33} = \boxed{36.94°}$

③ A BEAM OF UNPOLARIZED LIGHT TRAVELLING THROUGH THE AIR STRIKES A PANE OF GLASS AT 58°. THE REFLECTED BEAM IS POLARIZED. WHAT CAN WE CONCLUDE ABOUT INDEX OF THE GLASS?

$\theta = \theta_p \implies \theta = \tan^{-1} \frac{n_2}{n_1} \implies \tan\theta = \frac{n_2}{n_1}$

$n_2 = n_1 \tan\theta = (1) \tan 58° = \boxed{1.60}$